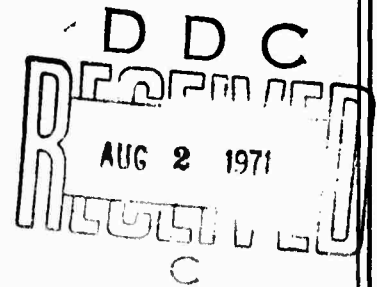


AD 727 103

United States Naval Postgraduate School



DEVELOPMENT OF A UNIFIED HEURISTIC MODEL
OF FLUID TURBULENCE

by

T. H. Gawain, D. Sc. and J. W. Pritchett

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February 1971

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122

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION	
Naval Postgraduate School Monterey, California (In cooperation with Naval Radiological Defense Laboratory, San Francisco)		Unclassified	
2b. GROUP			
3. REPORT TITLE			
DEVELOPMENT OF A UNIFIED HEURISTIC MODEL OF FLUID TURBULENCE			
4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)			
Technical Report 1971			
5. AUTHOR(S) (First name, middle initial, last name)			
Gawain, Theodore H. (Naval Postgraduate School *Pritchett, John W. (Naval Radiological Defense Laboratory)			
6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS	
February 1971	119	14	
8a. CONTRACT OR GRANT NO.	8b. ORIGINATOR'S REPORT NUMBER(S)		
ARPA Order No. 961, Program Code 7F40, Task 5, and NSSC Program SR 104 03 01 8. PROJECT NO.	NPS-57Gn71021		
c.	8b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
d.	NPS-57Gn9091A		
10. DISTRIBUTION STATEMENT			
This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY	
*Mr. Pritchett is now (Feb 71) with Information Research Associates Berkeley, California.		Advanced Research Projects Agency (ARPA), and Naval Ship Systems Command (NSSC) (Joint Sponsorship)	
13. ABSTRACT			
<p>A unified heuristic model of fluid turbulence is proposed which supplements the basic equations of motion and continuity so as to define determinate solutions under a wide variety of conditions. A single set of equations applies to all cases. Only the boundary conditions change for each application.</p> <p>The Reynolds stresses are related to the mean flow heuristically, through an eddy viscosity. This also involves the turbulent energy, which is analyzed with the aid of empirical expressions for dissipation and diffusion. A characteristic local length scale is introduced, based on a generalization of von Karman's mixing length.</p> <p>The model is illustrated by application to a channel and a jet. Agreement with experiment is satisfactory <i>note</i></p> <p>The report also gives the detailed rationale which underlies the model, including an original analysis of the stress/strain rate relation. Suggestions are made for further research.</p>			

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Fluid turbulence						
Scale of turbulence						
Eddy viscosity						
Turbulent jet						
Turbulent flow						
Poiseuille flow						
Spectrum of turbulence						
Correlation function						
Mixing length						
Turbulent dissipation						
Turbulent diffusion						
Turbulent energy						
Reynolds stresses						

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This task was supported by: Advanced Research Project Agency,
Order No. 961, Program Code 7F40, Task 5,
and Naval Ship Systems Command,
Program SR 1040301.

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ACKNOWLEDGEMENTS

The authors wish to express their appreciation to Dr. S. Lukasik, Deputy Director of the Advanced Research Project Agency, Department of Defense, for the interest and support which helped make this research possible, and to Mr. E. A. Schuert*, Director, Ocean Technology Branch, Naval Radiological Defense Laboratory, for the cooperation and encouragement which helped make it successful and rewarding.

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Part I

The Heuristic Model and Its Application

by

T. H. Gawain, D.Sc.

and

J. W. Pritchett

1.1 INTRODUCTION

The Naval Radiological Defense Laboratory, prior to its disestablishment in June 1969, had for some time been pursuing studies in numerical hydrodynamics in the field of underwater explosions. It is well known that fluid turbulence forms an important aspect of this problem. In fact, the lack of an adequate theory of turbulence has been an important limitation on the development of numerical hydrodynamics for applications not only to the explosion problem, but also to fluid mechanics problems of every type. There continues to be an urgent need to overcome this obstacle.

Consequently, an agreement was reached between NRDL and the Naval Postgraduate School some time ago for a joint attack on the problems of fluid turbulence. Two main approaches were investigated. The first approach emphasized fundamental understanding of the phenomena without undue concern about developing computational methods for application to practical problems. The second approach emphasizes the development of a feasible, heuristic computational method at the expense, if necessary, of analytical rigor. This report documents the excellent results that have been achieved in connection with this latter goal. Reference (1), although now largely superseded by more recent work, documents some earlier effort along the other avenue of attack. Further work is continuing at the Postgraduate School.

Unfortunately, the Naval Radiological Laboratory was disestablished in the midst of this research. The abrupt cessation of NRDL activities made it virtually impossible to complete certain details of the

mathematical model in a satisfactory manner. It was therefore decided to hold up distribution of a final report until such time as these additional points could be worked out. Meanwhile, the press of other sponsored research compelled the investigators to lay this task aside for a considerable time.

Fortunately, it was eventually possible to complete and revise the details mentioned above. Consequently, the present report, which describes the final model, is now being released for publication at last. Although the major part of the work described was accomplished some time ago, the model itself should continue to be of scientific interest for some time, owing to the basic and long term nature of the problem with which it deals.

As an interesting historic note, it seems worthwhile to mention that the heuristic model developed in this report resembles in some respects the approach first suggested by Prandtl in 1945, Ref. (2). It also has some similarities to, and some differences from, the more recent work of Harlow and his associates, References (3) and (4).

In incompressible flow, whether laminar or turbulent, the equations of momentum and continuity, along with the boundary conditions, theoretically suffice to establish completely the exact fluid motion. If the flow happens to be turbulent, however, the actual detailed motion becomes so complex that, although it is in principle determinate, its actual calculation would impose an overwhelming computational burden.

The usual way around this difficulty is to average the equations of momentum and continuity. This results, of course, in an enormous simplification. Unfortunately, it also involves a significant and irretrievable loss of essential information. Consequently, owing to the presence of the unknown Reynolds stresses which are created by this averaging process, the averaged equations of momentum and continuity do not in themselves comprise a determinate set.

In order to define a determinate solution, additional relations are required to fix the unknown Reynolds stresses. Unfortunately, these supplementary relations cannot be established from the original equations by any purely deductive process. For this purpose, supplementary empirical hypotheses are an unavoidable necessity.

From another viewpoint, it may be stated that the averaged equations of motion show the effect of the Reynolds stresses upon the mean flow. However, the reciprocal effect of the mean flow upon the Reynolds stresses is lost in the averaging process. Hence some adequate hypothesis must be found for representing this relation, at least approximately.

For this purpose, a heuristic approach which seems plausible is to postulate a relation between the Reynolds stresses and the mean flow which is analogous to the relation that is known to govern the viscous stresses. The analogue of the ordinary molecular kinematic viscosity is the so-called eddy kinematic viscosity. The problem becomes, therefore, to determine empirically the general law which governs this mean effective eddy viscosity at every space/time point in the flow field.

The eddy viscosity presumably depends on a number of variables, one of the most important of which is the local kinetic energy of turbulence. Therefore, it becomes necessary to find the space/time distribution of the turbulent energy. Fortunately, the governing energy equation can be deduced rigorously from the original equations of motion. However, the energy equation itself introduces two additional unknowns which can only be approximated in the same heuristic and empirical fashion as the eddy viscosity itself. The additional unknowns are the rate of dissipation of turbulent energy into heat, and the rate of turbulent diffusion of energy.

Theory and experiment both show that the eddy viscosity, and the dissipation and diffusion functions as well, depend not only on the

turbulent energy itself, but also on a local length scale parameter which can be associated with each space/time point in the flow field. Von Karman was perhaps the first to point out how a physically meaningful characteristic length can be defined in terms of local space derivatives of the mean velocity at any point in the flow. In the present paper, the original approach of von Karman is further developed and refined. It now takes into account not only the velocity derivatives at the designated point itself, but also the values in the general vicinity of the point.

By employing dimensional analysis, and by applying the available experimental data, we finally obtain three empirical expressions which determine to a reasonable approximation the eddy viscosity, the heat dissipation and the turbulent diffusion, respectively. These expressions also involve the turbulent energy, the local length parameter, and the distance to the nearest fixed wall (if any). Of course, these empirical expressions are amenable to further investigation and development.

In this way a single consistent and determinate set of equations is established which applies in principle to any incompressible turbulent flow field. Only the boundary conditions differ for each specific application.

The application of this heuristic model is shown for two widely different cases, namely, a two-dimensional duct, and an axi-symmetric jet. The distributions of energy and Reynolds stress are computed for each case and compared with experimental measurements. The agreement is satisfactory.

These results, although limited in extent, strongly suggest that the proposed heuristic model of fluid turbulence is basically correct, at least in its main essentials.

This report is divided into two parts. Part I summarizes the final heuristic model and illustrates its application to two typical cases, namely, a channel flow and a free jet. Part II gives the corresponding detailed background, theory and rationale. It tells why the final model is just what it is. It shows how the various parameters of the model are related to fundamental aspects of turbulent correlation and spectra. An original analysis is presented of the general tensor relation between stress and strain rate. This provides the basis for judging both the usefulness and the limitations of the eddy viscosity concept as employed in the heuristic model. The energy equation is derived from first principles, then transformed so as to be practically useful. This part of the report also raises various questions of a fundamental nature which could be fruitful for future research.

1.2 SYMBOLS

		<u>Ref. Eqs.</u>
a	Local dimensionless length scale in jet	1.6-31
a_1	Overall dimensionless length scale of cross-section in jet	1.6-28
A	Cross-sectional area	
b	Channel half width, or jet reference length	1.6-1
$A_1 A_2 A_3$ $B_1 B_2 B_3$	Functions in momentum equations for jet	1.6-9 thru -14
a, b, c, d k, l, m, n	Exponents in definitions of I^2 , J^2 , λ^2 , and τ^2	2.6-14 thru -19
C_λ, C_τ	Arbitrary constants in definitions of λ^2 and τ^2	2.6-8, -9
D	Determinant. Also diameter of jet	2.6-16
E	Mean kinetic energy of turbulence per unit mass	1.3-16, 2.3-5
E'	Instantaneous kinetic energy of turbulence per unit mass	A-12
\dot{E}_H	Rate of dissipation of turbulent energy per unit mass into heat.	2.8-8
f	Gaussian function	1.6-6
f_λ	Ratio of local length scale of mean flow to correlation length	2.5-1
f_τ	Ratio of local time scale of mean flow to correlation time	2.5-2
f_{ij}	Non-isotropic tensor in generalized stress/strain rate law	2.7-16
f_{ij}^*	Denotes values of f_{ij} for principal axes of strain rate	

F	Dimensionless stream function of jet	1.6-1
F'	Denotes derivative $\left(\frac{dF}{dn}\right)$	
F, G, H.	Functions in energy equation for two-dimensional channel	1.5-25, -26-27
g*	Parameter in stress/strain rate law	2.7-28
G ₁ , G ₂ , G ₃	Functions in analysis of jet	1.6-25, -26, -27
H	Integral of weighting function	1.3-11
H ^(S)	Integral of spatial weighting function	2.5-9
H ^(T)	Integral of temporal weighting function	2.5-10
i, j, k, l	Tensor indices	
I ² , J ²	Characteristic integrals for a point	1.3-13 and -14
l	Von Karman's mixing length	1.5-7, -8
(n)	Superscript indicating realization number	
N _R	Reynolds number	
p	Kinematic pressure (pressure divided by density) of turbulent fluctuations	
P	Kinematic pressure (pressure divided by density) of mean flow	
q	Root mean square turbulent velocity fluctuation	1.3-16
r, z, θ	Cylindrical coordinates: radial, axial, and angular, respectively	
R _{ij}	Correlation tensor	2.3-6
R _{ij} ^(S)	Spatial correlation tensor	2.3-9
R _{ij} ^(T)	Temporal correlation tensor	2.3-9
s	Exponential constant in Gaussian velocity distribution for jet (s = 0.102)	1.6-6
t	Time	

Δt	Time increment	2.3-1
T	Semi-interval for time averaging	2.2-4
t_{ij}	Viscous stress tensor for turbulent fluctuations	A-16
T_{ij}	Viscous stress tensor for mean flow	
T'_{ij}	Viscous stress tensor for mean flow plus turbulent fluctuation	A-2
u_i	Turbulent velocity fluctuation	2.2-2
U_i	Mean velocity	2.2-1
U'_i	Velocity of mean flow plus turbulent fluctuation	
U, V	Radial and axial components, respectively, of axi-symmetric mean flow	
U_o	Reference velocity of turbulent jet	1.6-1
$\left. \begin{matrix} -\overline{u_i u_j} \\ -\overline{uv} \end{matrix} \right\}$	Kinematic Reynolds stress tensor (stress divided by density)	2.7-1
v^*	Friction velocity	1.5-2
dv'	Element of volume	
W	Weighting function	1.3-10
$W^{(S)}$	Spatial weighting function	2.5-4
$W^{(T)}$	Temporal weighting function	2.5.5
w	Normalized weighting function	1.3-12
$w^{(S)}$	Normalized spatial weighting function	2.5-12
$w^{(T)}$	Normalized temporal weighting function	2.5-13
\bar{x}	Position vector of arbitrary fixed reference point	
\bar{x}'	Position vector of variable point	
$\Delta \bar{x}$	Displacement vector	1.3-9

x_1	} Cartesian coordinates	
x, y, z		
y	Distance to nearest wall, or distance from centerline of jet or center plane of channel	
z	Axial coordinate. Also nondimensional wall distance exponent	1.5-23
α	Dimensionless eddy viscosity coefficient	1.3-5,-18
β, β'	Dimensionless heat dissipation coefficients	1.3-19,2.8-10
γ	Dimensionless diffusion coefficient	1.3-20,2.8-21
γ_{ij}	Strain rate tensor of turbulent fluctuations	A-17
Γ_{ij}	Strain rate tensor of mean flow	1.3-4
$\Gamma_{11}^*, \Gamma_{22}^*, \Gamma_{33}^*$	Strain rate tensor components for principal axes of strain rate	
δ_{ij}	The Kronecker delta tensor, with value unity for equal indices, but zero otherwise	
ϵ	Effective eddy kinematic viscosity	2.7-22
ϵ_0	Eddy viscosity in centerline of jet	1.6-16
$\epsilon_{jk}, \epsilon_{ijkl}$	Tensor versions of eddy viscosity	2.7-13,-14
ϵ''	"Pressure diffusion" coefficient	2.8-17
ϵ'	Kinetic energy diffusion coefficient	2.8-18
ζ	Radial coordinate	B-11
η	Dimensionless distance from center plane of two-dimensional channel, or from centerline of jet	1.5-1
θ, θ'	Angular coordinate	
θ^*	Strain rate parameter in stress/strain rate law	2.7-26
κ	Constant in von Karman's mixing length theory	1.5-8

$\bar{\kappa}, \kappa_1$	Wave number vector	1.5-8
λ	Local length scale of mean flow	1.3-15
λ_1	Overall length scale	1.3-8
λ^*	Correlation distance	2.3-11
λ_D	Dissipation length	2.8-8
ν	Molecular kinematic viscosity	
ξ	Separation distance in meridional plane	B-6
Σ	Summation sign	
τ	Local time scale of mean flow	2.6-18
τ^*	Correlation time	2.3-12
τ'_{ij}	Kinematic Reynolds stress tensor (stress divided by density)	2.7-1
τ'_{ij}	Distortional kinematic stress tensor	2.7-4
τ''_{ij}	Residual non-isotropic kinematic stress tensor	2.7-19
$(\tau'_{ij})_v$	Viscous distortional stress tensor	2.7-10
τ'^2	Invariant of the tensor τ'_{ij}	2.7-5
τ''^2	Invariant of the tensor τ''_{ij}	2.7-20
ϕ_{ij}	Fourier transform of the correlation tensor R_{ij}	2.4-2
ψ	Stream function for jet. Also auxiliary weighting function for axi-symmetric flows	1.6-1 B-13
ψ^*	Stress parameter in stress/strain rate law	2.7-29
ω	Angular frequency or temporal wave number	
Ω	Invariant of strain rate tensor representing generalized shearing rate	1.3-6
Ω'	Absolute magnitude of the gradient of Ω	1.3-7, 2.6-5
Note: In the jet analysis, subscript 1 on any quantity denotes the values at $z = 1$. It is therefore always a function of η only.		

1.3 SUMMARY OF BASIC EQUATIONS OF THE HEURISTIC MODEL

The purpose of this section is to summarize the basic equations which, taken together, constitute the proposed heuristic model of fluid turbulence. A few explanatory comments are offered to identify these various equations and to show their general relation to the overall theory, but no detailed justification is given in this section; such details are included in other parts of this report. Also, the equations in this section are stated in three dimensional cartesian form. The reductions to forms specifically applicable to plane or axi-symmetrical mean flows are treated elsewhere.

An incompressible mean flow field must everywhere satisfy the continuity condition

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1.3-1)$$

and the ensemble averaged Navier Stokes equations of motion

$$\left(\frac{\partial u_i}{\partial t} \right) + \frac{\partial}{\partial x_j} (u_i u_j) = \nu \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right) - \left(\frac{\partial p}{\partial x_i} \right) + \left(\frac{\partial \tau_{ij}}{\partial x_j} \right) \quad (-2)$$

as well as the particular boundary conditions appropriate to the specific flow field under consideration.

However, because of the occurrence of the unknown Reynolds stress τ_{ij} , Eqs. (-1) and (-2) do not in themselves constitute a determinate set of equations. In order to define a determinate solution, additional relations are required. Purely deductive theory alone does not provide these relations.

Hence appropriate supplementary hypotheses must be tentatively postulated, subject to eventual verification, modification or disproof by experimental evidence.

For the reason discussed at length elsewhere, we postulate that the Reynolds stresses can be adequately related to the strain rates of the mean flow through a law which has been deliberately somewhat oversimplified to the form

$$\tau_{ij} = -\frac{1}{3} q^2 \delta_{ij} + \epsilon \Gamma_{ij} \quad (-3)$$

In this equation, q^2 is the mean square velocity of the turbulent fluctuations, ϵ is the so-called eddy kinematic viscosity, and Γ_{ij} is the strain rate of the mean flow as defined by the expression

$$\Gamma_{ij} = \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (-4)$$

Unfortunately, the eddy viscosity ϵ as defined above is not a simple property of the fluid itself, but rather a complex property of the turbulent flow field. Physical and dimensional reasoning leads to a supplementary postulate, namely that ϵ is expressible in the following form.

$$\left(\frac{\epsilon}{\lambda q} \right) = \alpha \quad (-5)$$

Here λ represents a local length scale, defined below, which varies from point to point in the field, but which is everywhere related in a consistent way to the local mean velocity distribution. The dimensionless coefficient α is a slowly varying universal function not predictable from theory, but determinable from experimental data. This function is specified later in Eq. (-18)

To establish the quantity λ at any point \bar{x} , we proceed as follows. We first compute the following mean flow variables throughout the field, namely,

$$\Omega^2 = \frac{1}{2} r_{ij} r_{ij} \quad (-6)$$

$$(\Omega')^2 = \frac{1}{4} \left(\frac{\partial \Omega^2}{\partial x_i} \right) \left(\frac{\partial \Omega^2}{\partial x_i} \right) \quad (-7)$$

Then we compute the reference quantity

$$\lambda_1^2 = \frac{\int \Omega^4 dv}{\int (\Omega')^2 dv} \quad (-8)$$

where the volume integrals extend over the entire field. Of course λ_1^2 is therefore a characteristic constant for the field as a whole and is independent of \bar{x} , but it may vary as a function of time if the mean flow be unsteady.

The calculation continues as follows. All integrals extend over the entire field of flow. Let

\bar{x} = coordinate of the fixed point at which λ is being evaluated

\bar{x}' = coordinate of variable point at which integrands below are evaluated

$$\Delta \bar{x} = (\bar{x}' - \bar{x}) = \text{separation variable} \quad (-9)$$

$$W = e^{-\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}} = \text{weighting function} \quad (-10)$$

$$H = \int W dv' = \text{normalizing integral} \quad (-11)$$

$$w = \frac{W}{H} = \text{normalized weighting function} \quad (-12)$$

$$I^2 = \int \omega \Omega^4 dv' = \text{first characteristic integral} \quad (-13)$$

$$J^2 = \int \omega (\Omega \Omega')^2 dv = \text{second characteristic integral} \quad (-14)$$

Of course the integrals I^2 and J^2 are functions of \bar{x} and, if the mean flow is unsteady, of time t as well.

The required length scale λ at any point \bar{x} is finally given by the ratio

$$\lambda^2 = \frac{I^2}{J^2} \quad (-15)$$

The above calculation sequence establishes not only the desired local length scale $\bar{\lambda}$ but also the useful auxiliary quantity J . It turns out that J is needed below in connection with the energy equation. The rationale involved is fully explained elsewhere in this report.

If the mean flow happens to be either plane or axi-symmetric, all of the above volume integrals can be reduced to corresponding surface integrals. The reduction for the axi-symmetric case is rather interesting; it is given in Appendix B.

If the flow field happens to be self-similar, the above calculations need be made on only one representative cross-section. Corresponding results for other cross-sections are then found simply by applying the proper scaling rules. If the flow field is not self-similar, all points must be computed independently. If the flow field is steady, the calculations can be made once and for all. If it is unsteady, they must, of course, be repeated at appropriate intervals of time.

Once the distribution of λ has finally been established, it is then necessary to determine the distribution of the turbulent energy E , or equivalently, of the root mean square turbulent velocity

$$q = \sqrt{2E} \quad (-16)$$

This is governed by the energy equation

$$\left(\frac{\partial E}{\partial t}\right) = \epsilon \Omega^2 - \beta (2E)^{7/6} J^{1/3} - \frac{\partial}{\partial x_1} [U_1 E] + \frac{\partial}{\partial x_1} \left[\gamma \epsilon \left(\frac{\partial E}{\partial x_1} \right) \right] \quad (-17)$$

Of course the eddy viscosity ϵ in the energy equation is the same as that earlier defined in Eq. (-5) in connection with the stress/strain rate law. If desired, Eq. (-5) may be used to eliminate ϵ from the energy equation. In any case, the coefficient α which governs ϵ is fixed by Eq. (-18) below.

Each term in Eq. (-17) has dimensions of time rate of change of energy per unit mass. The four terms on the right represent, respectively, generation of turbulent energy, dissipation to heat, convection by the mean flow, and turbulent diffusion. Molecular diffusion is negligible.

The dimensionless coefficients β and γ which occur in the energy equation, like the dimensionless coefficient α introduced earlier, are assumed to be well behaved universal functions. However, they are not determinable from theory, but must be established empirically.

On the basis of the work done so far, the following expressions seem to provide satisfactory agreement with the available data.

$$\alpha = 0.065 \left\{ 1 + e^{-\left(\frac{y}{\lambda} - 1\right)^2} \right\} \quad (-18)$$

$$\frac{1}{\beta} = 3.7 \left\{ 1 + e^{-\left(\frac{y}{\lambda} - 1\right)^2} \right\} \quad (-19)$$

$$\gamma = 1.4 - 0.4 e^{-\left(\frac{y}{\lambda} - 1\right)^2} \quad (-20)$$

In these expressions, y is the distance to the nearest fixed boundary. As the boundary is approached y approaches zero, but it turns out that under these conditions, the length parameter λ also approaches zero simultaneously, so that the ratio $\frac{y}{\lambda}$ remains finite. In fact, it happens that at the wall itself, $\left(\frac{y}{\lambda}\right)$ equals unity (not zero). In any case, it is stipulated in connection with the above formulas, that $\left(\frac{y}{\lambda}\right)$ shall arbitrarily be assigned a lower limit of unity. On the other hand, far from any fixed boundary, or in the absence of fixed boundaries, we set $y = \infty$, whereupon α , β , and γ reduce to three simple constants.

The proposed heuristic model of fluid turbulence consists, therefore, of the foregoing equations (-1) through (-20) inclusive, plus the particular boundary conditions appropriate to the specific problem under consideration. This system of equations theoretically defines a determinate solution for the general case of inhomogeneous and non-stationary fluid turbulence. It fixes the mean flow field, the distribution of turbulent energy and the Reynolds stresses.

The results obtained so far tend to support the proposed model, but further work is necessary to ascertain more fully its adequacy and range of applicability. Various features of the model are readily amenable to modification if necessary, so as to improve its range and accuracy.

An important merit of the model, if it continues to prove successful, is that it reduces a wide range of dissimilar flow geometries and flow conditions to a single unified theory.

1.4 GENERAL METHOD OF SOLUTION

Although the equations (-1) through (-20) of the previous section, along with the appropriate boundary conditions, theoretically suffice to

determine all characteristics of the flow field, the actual solution of this system of equations for any non-trivial case is by no means simple.

Let us consider first the general case of an unsteady mean flow, with the initial velocity distribution of time $t = 0$ arbitrarily prescribed.

Of course, the prescribed initial velocity distribution must satisfy the continuity condition, Eq. (-1). This offers no particular difficulty. In the usual case in which the mean flow is either plane or axis-symmetric, the continuity condition may be satisfied identically by expressing the velocity components in terms of a stream function. Let us suppose that this has been done, and that the corresponding mean flow velocity components at time $t=0$ have been calculated. The subsequent calculation procedure may now be described in somewhat general terms as follows.

The next step is to perform the sequence of calculations summarized in Eqs. (-4) through (-15). In this way is obtained the local length scale λ at every point in the field, and the quantity J .

Next we must solve the energy equation (-17) for the mean square turbulent velocity q at every field point. For this purpose we substitute Eq. (-5) into (-17) and use Eqs. (-18), (-19), and (-20) to evaluate the three empirical coefficients α , β , and γ . (In some cases the actual distribution of the turbulent energy $E = \frac{q^2}{2}$ may be known from experimental measurement. If so, we have at this point one important comparison between theory and experiment.)

Once estimates of λ and q are available from the foregoing calculations, it becomes a simple matter to solve Eq. (-5) for the eddy viscosity ϵ , and Eq. (-3) for the Reynolds stresses.

The Reynolds stresses obtained in this way may now be substituted back into the Navier Stokes relations, Eqs. (-2). By taking the divergence of Eqs. (-2), we find that

$$\left(\frac{\partial^2 P}{\partial x_i \partial x_j} \right) = - \frac{\partial^2}{\partial x_i \partial x_j} [U_i U_j + \tau_{ij}] \quad (1.4-1)$$

Since mean velocity and stress are at this point known, the above result can be solved for the pressure distribution, subject to the specified boundary conditions on pressure. This part of the solution involves a lengthy iteration procedure, but is straightforward in principle.

Once U_i , τ_{ij} , and P are all established, we can revert to Eq. (-2) and solve for the time rate of change of velocity $\left(\frac{\partial U_i}{\partial t} \right)$ at every field point. By integrating forward through time over a small interval, we can find the new mean velocity distribution at the later time.

The above procedure may be repeated as often as desired, thus tracing the gradual evolution of the flow field over time. If the boundary conditions happen to be stationary in time, the entire flow field will evolve toward a stationary state, thus defining the solution of the corresponding steady state problem.

The above process of reaching a steady state asymptotically in time can be quite expensive in terms of computational effort and time required. It is often possible to shorten or abridge the above calculation in various ways, depending on the situation. For example, instead of substituting the Reynolds stresses predicted by Eq. (-3) into the Navier Stokes law Eq. (-2), we can solve the latter for the Reynolds stresses "required" in order to maintain the prescribed mean motion in a steady state. These "required" stresses can then be compared with the stresses "predicted" from Eq. (-3). If the agreement between the two sets of

stresses is deemed satisfactory, this can be taken as a verification of the adequacy of the given velocity distribution as a estimate of the steady state. This is the procedure that has been followed in our sample problems, and it has proved entirely satisfactory for the intended purpose. In this case the purpose has been to demonstrate the degree of agreement of our heuristic model with experimental observations. The agreement is considered satisfactory for both cases analyzed.

1.5 APPLICATION TO A TWO-DIMENSIONAL CHANNEL

Consider the steady, incompressible, turbulent flow through a two-dimensional channel of uniform height $2b$ as shown in Fig. 1.5-1. Let y be distance from the channel center-line. Choosing b as the reference unit of length, the corresponding dimensionless distance becomes

$$\frac{y}{b} = \eta \quad (1.5-1)$$

Let the kinematic shear stress at the wall be

$$\tau_1 = v_*^2 \quad (-2)$$

where v_* is the so-called friction velocity. Without loss of generality, we can always choose reference units of time such that

$$\tau_1 = v_*^2 = 1 \quad (-3)$$

The kinematic shear stress τ varies linearly across the channel so that at any location η we have

$$\tau = \eta \quad (-4)$$

According to our present notation, if U is the mean velocity at any η , then

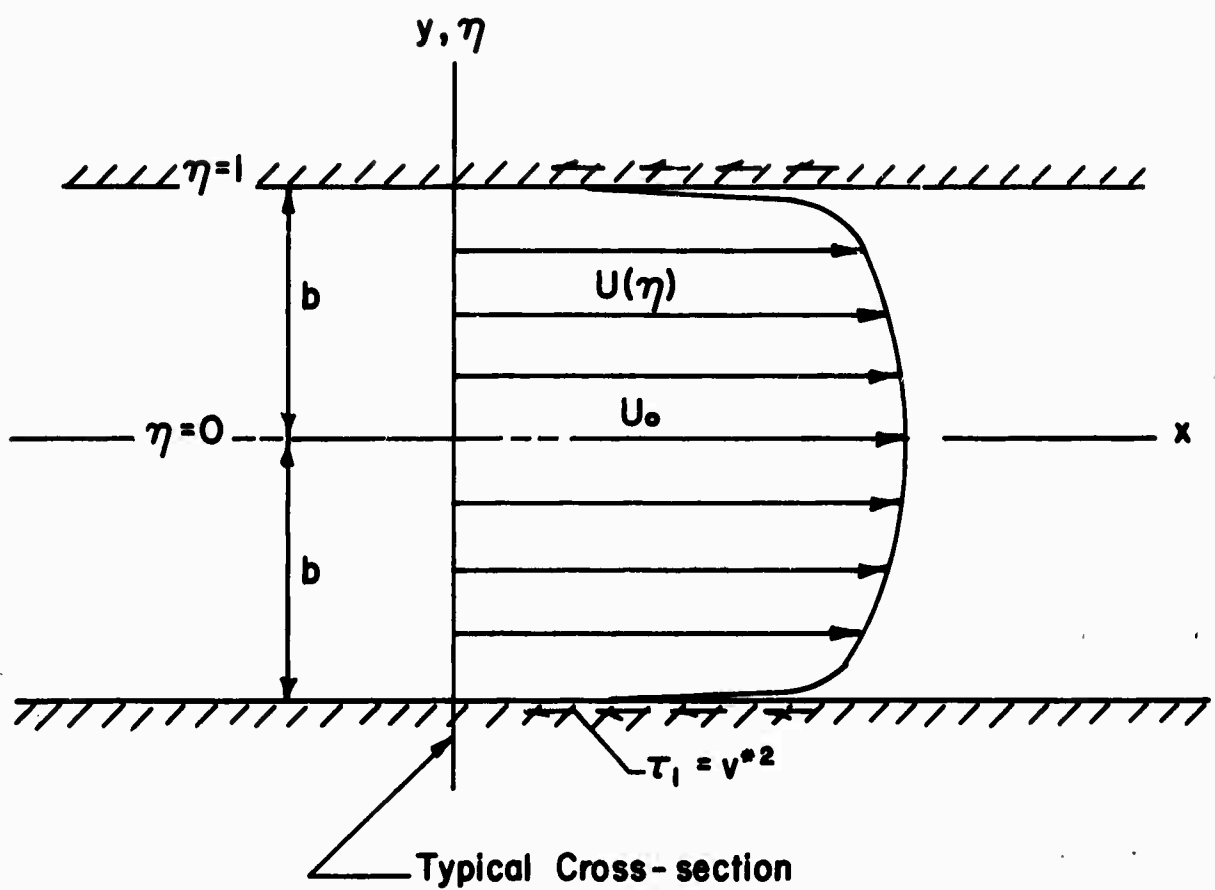


FIGURE 1.5-1 FLOW IN A TWO-DIMENSIONAL CHANNEL

$$\left(\frac{dU}{d\eta}\right) = U' = \Omega \quad (-5)$$

$$\left(\frac{d^2U}{d\eta^2}\right) = U'' = \Omega' \quad (-6)$$

According to the von Karman mixing length theory

$$\tau = \ell^2 \Omega^2 \quad (-7)$$

where the mixing length ℓ is given, in our notation, by

$$\ell^2 = \kappa^2 \frac{\Omega^2}{\Omega'^2} \quad (-8)$$

and where κ is a constant which remains to be determined.

Upon combining Eqs. (-4), (-7), and (-8), taking the square root of both sides and rearranging, we obtain the basic relation

$$\frac{\Omega'}{\Omega^2} = \frac{\kappa}{\sqrt{\eta}} \quad (-9)$$

This equation is readily integrable and gives the well known von Karman velocity profile which in the present notation becomes

$$\frac{U}{v_*} = U = U_0 + \frac{1}{\kappa} \{ \ln(1 - \sqrt{\eta}) + \sqrt{\eta} \} \quad (-10)$$

It is known that this results gives satisfactory agreement with experiment if we place

$$\kappa = 0.36 \quad (-11)$$

The resulting quantities of interest in the present analysis are

$$\Omega = U' = - \frac{1}{2\kappa(1 - \sqrt{\eta})} \quad (-12)$$

and

$$\Omega' = U'' = - \frac{1}{4\kappa \eta(1 - \sqrt{\eta})^2} \quad (-13)$$

We next require the "averaging distance" λ_1 defined by the equation

$$\lambda_1^2 = \lim_{\eta_1 \rightarrow 1} \frac{\int_0^{\eta_1} \Omega^4 d\eta}{\int_0^{\eta_1} (\Omega\Omega')^2 d\eta} \quad (-14)$$

Upon substituting expressions (-12) and (-13) into (-14), we find that in the limit as η_1 approaches unity, both integrals approach infinity, but the above ratio of these integrals approaches zero. Hence we obtain the unusual result that, in this case,

$$\lambda_1^2 = 0 \quad (-15)$$

Now the local length scale λ may be computed from the relation

$$\lambda^2 = \lim_{\lambda_1 \rightarrow 0} \frac{\int_0^1 \frac{1 - \frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}{e^{\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}} \Omega^4 d\eta'}{\int_0^1 \frac{1 - \frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}{e^{\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}} (\Omega\Omega')^2 d\eta'} = \frac{\Omega^2}{\Omega'^2} \quad (-16)$$

This result is remarkable! It shows that our characteristic length λ is in this case proportional to the classical von Karman mixing length l . Form Eqs. (-8), (-12), (-13) and (-16) we find that

$$\lambda = \frac{l}{\kappa} = \frac{\Omega}{\Omega'} = 2\sqrt{\eta}(1 - \sqrt{\eta}) \quad (-17)$$

Similarly, because of the exceptional result (-15), the integral for J^2 reduces simply to

$$J^2 = \lim_{\lambda_1 \rightarrow 0} \frac{\int_0^1 \frac{1 - \frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}{e^{\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}} (\Omega\Omega')^2 d\eta'}{\int_0^1 \frac{1 - \frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}{e^{\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}}} d\eta'} = (\Omega\Omega')^2 \quad (-18)$$

Therefore again using (-12) and (-13) we find that

$$J = \Omega\Omega' = \frac{1}{8\kappa^2\sqrt{\eta}(1 - \sqrt{\eta})^3} \quad (-19)$$

The three empirical functions α , β , and γ may now be expressed in the form

$$\frac{\epsilon}{\lambda\sqrt{2E}} = \alpha = 0.065\{1 + e^{-z^2}\} \quad (-20)$$

$$\frac{1}{\beta} = 3.7\{1 + e^{-z^2}\} \quad (-21)$$

and

$$\gamma = 1.4 - 0.4 e^{-z^2} \quad (-22)$$

where

$$z^2 = \frac{1}{4}\left(\frac{1}{\sqrt{\eta}} - 1\right)^2 \quad (-23)$$

With the functions Ω , λ , J , α , β , and γ now known, we can proceed to solve the energy equation. The unsteady term and the convective term both vanish from this equation in the present instance and we obtain

$$\alpha\lambda\sqrt{2E}\Omega^2 - \beta(2E)^{7/6}J^{1/3} + \frac{\partial}{\partial\eta}\left[\alpha\gamma\lambda\sqrt{2E}\left(\frac{\partial E}{\partial\eta}\right)\right] = 0 \quad (-24)$$

Energy E is now the only unknown. To simplify the equation further, we introduce the following notation.

$$\alpha\lambda\Omega^2 = F(\eta) \quad (-25)$$

$$\beta J^{1/3} = G(\eta) \quad (-26)$$

$$\alpha\gamma\lambda = H(\eta) \quad (-27)$$

Of course, these three functions are now known. Therefore the energy equation reduces to

$$\frac{d}{d\eta} \left[H \sqrt{2E} \left(\frac{dE}{d\eta} \right) \right] - G(2E)^{7/6} + F \sqrt{2E} = 0 \quad (-28)$$

This equation cannot be solved analytically, but it can easily be expanded and rearranged for numerical solution by finite differences.

The desired boundary conditions are

$$\text{At } \eta = 0 \quad \left(\frac{dE}{d\eta} \right)_0 = E'_0 = 0 \quad (-29)$$

$$\text{At } \eta = 1 \quad E_1 = 0 \quad (-30)$$

This last condition cannot be satisfied directly. However, we can integrate from the centerline out, using a trial value of the centerline energy E_0 . By trial and error, a value of E_0 can then be found which will in fact yield the required condition $E_1 = 0$ at the wall.

This is the method which was used to compute the energy distribution shown by the solid line, in Fig. 1.5-2. It may be seen that the curve obtained in this way is in reasonably good agreement with the experimental data of Reichart and of Laufer, Refs. 5 and 6.

Once the energy distribution is known, the dimensionless Reynolds stresses may be computed from the relation

$$-\overline{uv} = \tau = \alpha \lambda \sqrt{2E} \Omega \quad (-31)$$

Except very near the walls, the above Reynolds stress should agree with the total shear stress defined by Eq. (-4).

If we wish to switch back to absolute units, Eqs. (-4) and (-31) reduce respectively to

$$\frac{\tau}{v_*^2} = \frac{y}{b} \quad \text{Total Stress} \quad (-32)$$

$$-\frac{\overline{uv}}{v_*^2} = \alpha \left(\frac{\lambda}{b} \right) \left(\frac{\sqrt{2E}}{v_*} \right) \left(\frac{b\Omega}{v_*} \right) \quad \text{Reynolds Stress} \quad (-33)$$

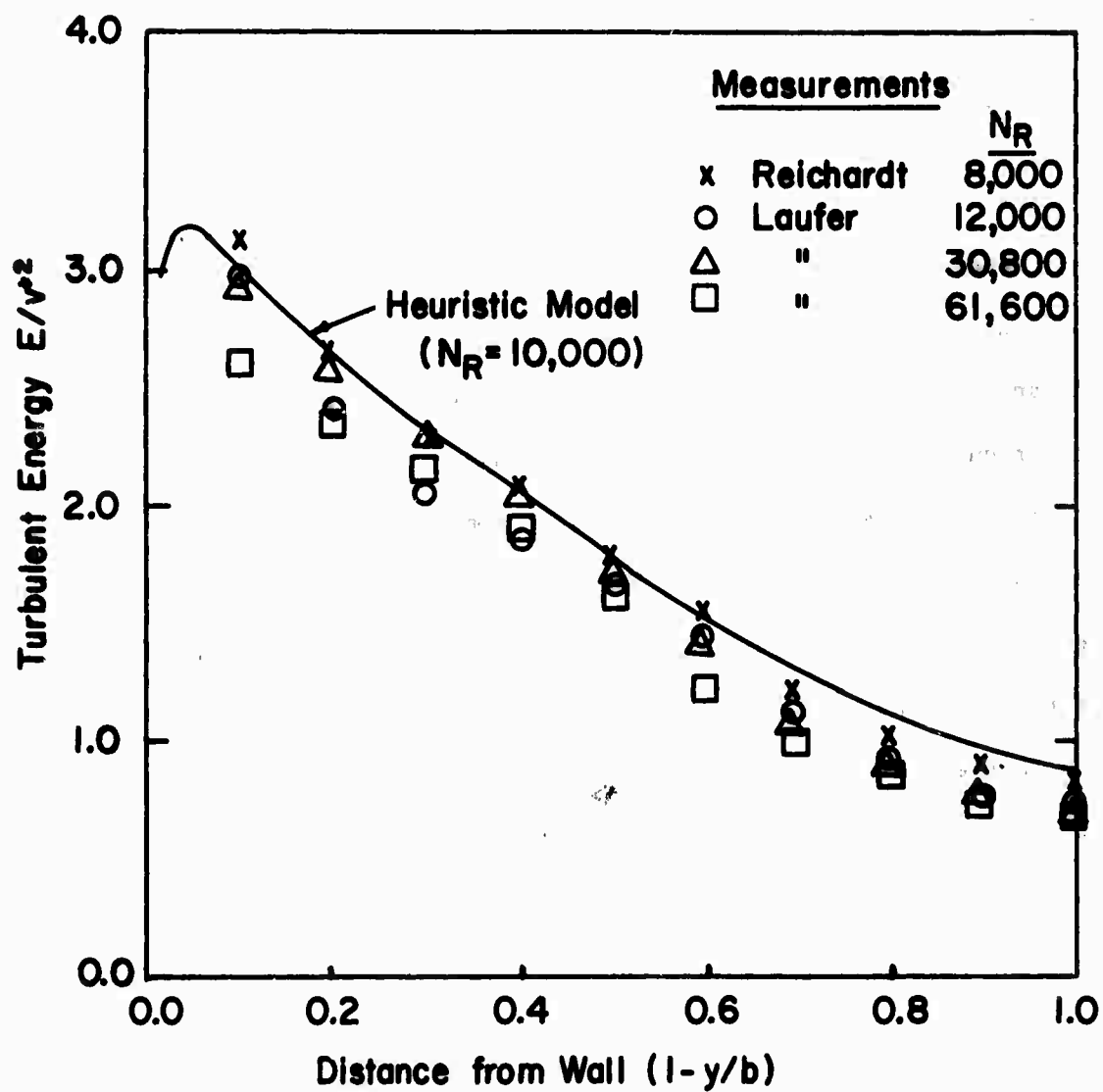


FIGURE I.5-2 TURBULENT ENERGY DISTRIBUTION IN A TWO DIMENSIONAL CHANNEL

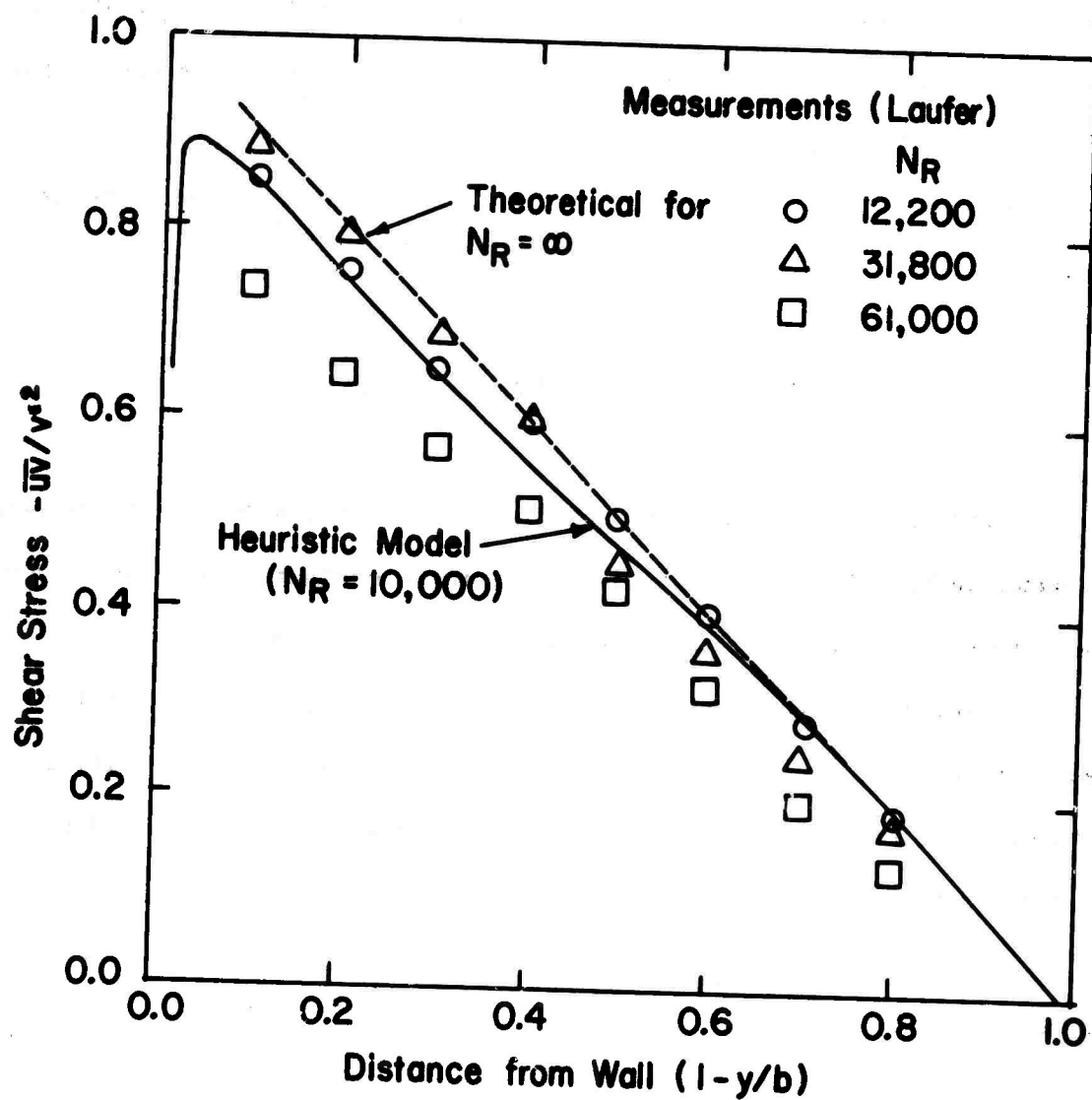


FIGURE I.5-3 REYNOLDS SHEAR STRESS DISTRIBUTION IN A TWO DIMENSIONAL CHANNEL

In Fig. 1.5-3, the solid line shows the above heuristic estimate of the Reynolds stress, computed for a Reynolds number of 10,000. The dashed straight line represents the true total stress, which should agree closely with the Reynolds stress everywhere except in the immediate vicinity of the wall. The agreement is seen to be good; the computed results are on the whole more accurate than are the experimental measurements from Reference (6), as shown on this figure.

We conclude that this example tends to substantiate the proposed heuristic model of turbulence.

1.6 APPLICATION TO AN AXI-SYMMETRIC JET

Consider a free turbulent jet discharging into a quiescent atmosphere as shown in Fig. 1.6-1. The radial and axial coordinates are r and z and the corresponding velocity components are U and V , respectively.

Experimental results indicate that the velocity profiles of the mean flow at various cross-sections are self-similar. Specifically, if z be measured from a suitable virtual origin, the stream function ψ can be reduced to the form

$$\psi = U_0 b z F(\eta) \quad (1.6-1)$$

where

$$\eta = \frac{r}{z} \quad (-2)$$

and where U_0 and b are constants of the jet.

Consequently the velocity components of the mean flow become

$$U = -\frac{1}{r} \left(\frac{\partial \psi}{\partial z} \right)_r = \left(\frac{U_0 b}{z} \right) \left(F' - \frac{F}{\eta} \right) \quad (-3)$$

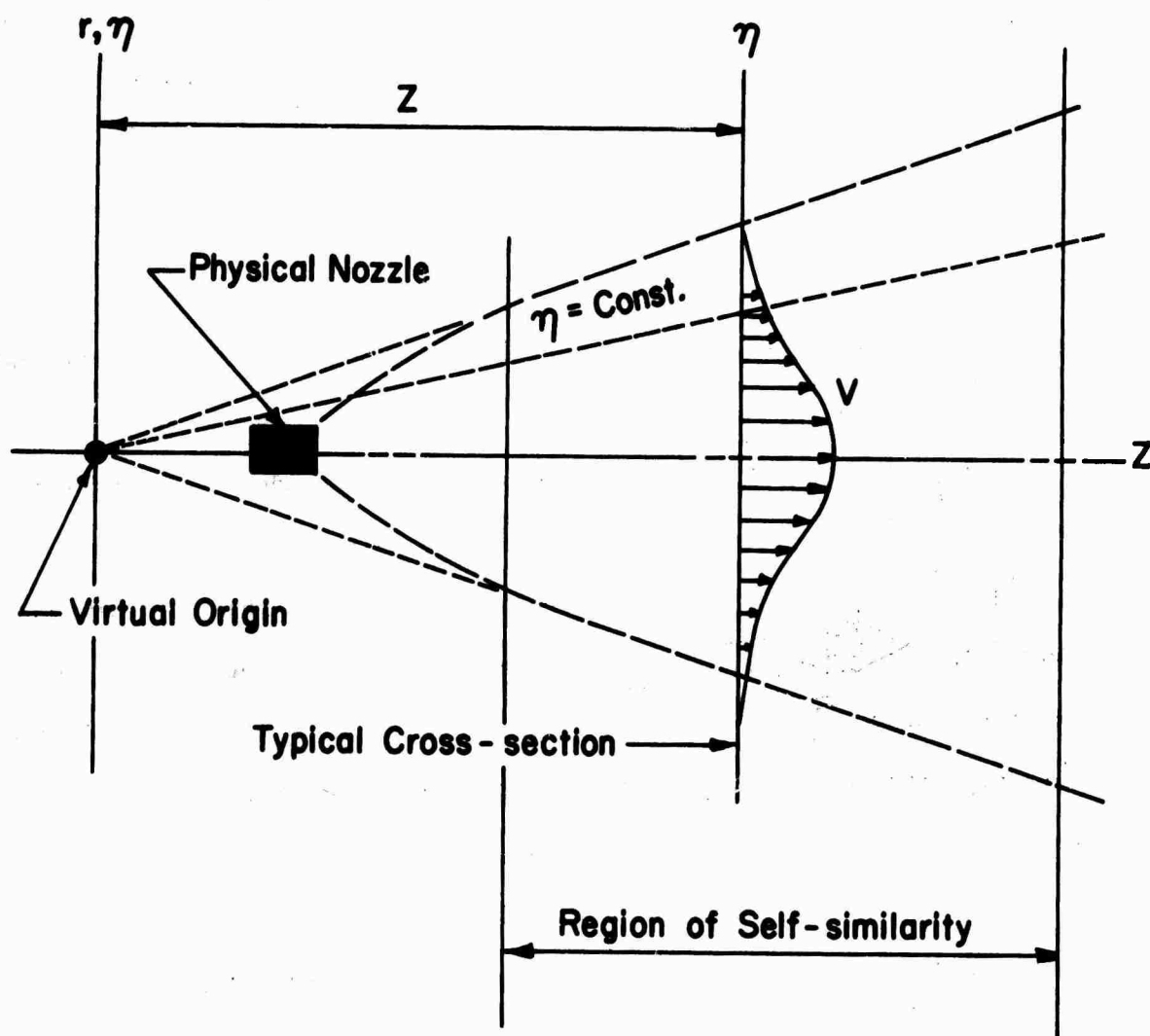


FIGURE I.6-1 THE AXI-SYMMETRIC JET

$$v = + \frac{1}{r} \left(\frac{\partial \psi}{\partial r} \right)_z = \left(\frac{U_0 b}{z} \right) \left(\frac{F'}{\eta} \right) \quad (-4)$$

where

$$F' = \left(\frac{\partial F}{\partial \eta} \right)_z \quad (-5)$$

The known experimental results also indicate that the generalized velocity distribution through the jet is very well approximated by the simple expression

$$\frac{F'}{\eta} = f = e^{-\frac{\eta^2}{s^2}} \quad (-6)$$

Where s is a characteristic constant for a turbulent jet. It has the known value $s = 0.102$.

Furthermore, it is useful and convenient to adopt units of length and time such that, when expressed in these units, $U_0 = 1$ and $b = 1$. Of course, there is no loss in generality involved in this. It amounts merely to non-dimensionalizing all quantities with respect to U_0 and b as reference parameters. Hence symbols like ϵ , p , u , v and so on will now represent the corresponding dimensionless versions of kinematic viscosity, kinematic pressure, velocity components and the like.

The next step in the analysis is to combine the momentum equations (-2) and the stress law (-3) of section 1.3. The results may be transformed to the z , η coordinate system defined above. The foregoing dimensionless functions ϵ and p may be introduced therein, and the results reduced. This reduction, although very lengthy, is straightforward. In this way the following results are obtained.

$$\eta p' + 2p + A_1(\eta)\epsilon' + A_2(\eta)\epsilon = A_3(\eta) \quad (-7)$$

$$-p' + 0 + B_1(\eta)\epsilon' + B_2(\eta)\epsilon = B_3(\eta) \quad (-8)$$

The A's and B's are known functions defined as follows, where f is the Gaussian function defined by Eq. (-6).

$$A_1(\eta) = \left(1 - \frac{2}{s^2} - \frac{2\eta^2}{s^2}\right) \eta f \quad (-9)$$

$$A_2(\eta) = \left[\left(2 - \frac{4}{s^2}\right) - \left(5 - \frac{2}{s^2}\right) \frac{2\eta^2}{s^2} + \frac{4\eta^4}{s^4} \right] f \quad (-10)$$

$$A_3(\eta) = f(1 - 2f) \quad (-11)$$

$$B_1(\eta) = \left(1 - \frac{2}{s^2} - \frac{2\eta^2}{s^2}\right) \eta^2 f + \frac{s^2}{\eta} (1 - f) \quad (-12)$$

$$B_2(\eta) = \left[3 \left(1 - \frac{2}{s^2}\right) - \left(3 - \frac{1}{s^2}\right) \frac{4\eta^2}{s^2} + \frac{4\eta^4}{s^4} \right] \eta f \quad (-13)$$

$$B_3(\eta) = -\eta f^2 + \left(\eta + \frac{s^2}{2\eta}\right) f(1 - f) - \frac{s^4}{4\eta} (1 - f)^2 \quad (-14)$$

Theoretically, it is possible to eliminate p and p' between the two momentum equations (-7) and (-8). This yields a single vorticity equation of second order in ϵ which can be solved for the unknown eddy viscosity distribution $\epsilon(\eta)$.

However, it is known from experiment that the terms in p and p' are negligible in Eqs. (-7) and (-8). Moreover, the terms in Eq. (-8) are negligible in comparison with those in Eq. (-7). We can take advantage of this information to simplify the solution for $\epsilon(\eta)$. We simply discard Eq. (-8), and drop the terms in p and p' from Eq. (-7). The resulting equation is now only of first order in ϵ . It may be written in the form

$$\left(\frac{d\epsilon}{d\eta}\right) + \left\{ \frac{\left(2 - \frac{4}{s^2}\right) - \left(5 - \frac{2}{s^2}\right) \frac{2\eta^2}{s^2} + \frac{4\eta^4}{s^4}}{\left(1 - \frac{2}{s^2} + \frac{2\eta^2}{s^2}\right) \eta} \right\} \epsilon = \frac{(1 - 2f)}{\left(1 - \frac{2}{s^2} - \frac{2\eta^2}{s^2}\right) \eta} \quad (-15)$$

Solution of this equation analytically is not feasible, but solution by numerical integration is simple.

Note first that at the centerline $\eta = 0$, this equation gives at once

$$\epsilon_0 = \frac{1}{2 \left(\frac{2}{s^2} - 1 \right)} = 0.00260 \quad (-16)$$

This result provides the boundary value required for the numerical integration.

It is also of interest to note that for very large values of η , Eq. (-15) simplifies to the following approximation, namely,

$$\left(\frac{d\epsilon}{d\eta} \right) - \left(\frac{2\eta}{s^2} \right) \epsilon = - \left(\frac{s^2}{2\eta^3} \right) \quad (-17)$$

The solution is

$$\epsilon = \frac{s^5}{2} e^{+s\eta^2} \int_{\eta}^{\infty} \eta^{-5} e^{-s\eta^2} d\eta \quad (-18)$$

The true numerical solution approaches the above analytical curve asymptotically at large values of η .

With the dimensionless eddy viscosity distribution $\epsilon(\eta)$ known, it becomes a simple matter to find the corresponding dimensionless Reynolds shear stress τ_{rz} from the relation

$$\tau_{rz} = -\overline{uv} = \epsilon \left[\left(\frac{\partial U}{\partial z} \right) + \left(\frac{\partial V}{\partial r} \right) \right] \quad (-19)$$

By making use of Eqs. (-3) through (-6) this result can be reduced to the following form for computation

$$\tau_{rz} = -\overline{uv} = \frac{\epsilon(\eta)}{z^2} \left\{ - \left(3 + \frac{2}{s} \right) \eta + \frac{2}{s^2} \eta^3 \right\} f \quad (-20)$$

The shear stress computed from this equation is shown by the dashed line in Fig (1.6-2).

To form the mean flow parameters required for the heuristic model, we note that, in cylindrical coordinates,

$$\Omega^2 = 2 \left(\frac{\partial U}{\partial r} \right)^2 + 2 \left(\frac{\partial V}{\partial z} \right)^2 + 2 \left(\frac{U}{r} \right)^2 + \left[\left(\frac{\partial U}{\partial z} \right) + \left(\frac{\partial V}{\partial r} \right) \right]^2 \quad (-21)$$

and

$$(\Omega\Omega')^2 = \frac{1}{4} \left[\left(\frac{\partial \Omega^2}{\partial r} \right)^2 + \left(\frac{\partial \Omega^2}{\partial z} \right)^2 \right] \quad (-22)$$

The change to the new coordinates η, z is quite lengthy, but not difficult otherwise. The results finally obtained can be summarized in the following form.

$$\Omega^2 = \frac{1}{z} G_1(\eta) \quad (-23)$$

$$(\Omega\Omega')^2 = \frac{1}{z^{10}} \left[G_2^2(\eta) + G_3^2(\eta) \right] \quad (-24)$$

where the G 's are known functions defined as follows.

$$G_1(\eta) = \left[12 + \left(\frac{4}{s} + \frac{12}{s^2} - 15 \right) \frac{\eta^2}{s^2} - \frac{4}{s^2} \left(\frac{2}{s^2} - 3 \right) \eta^4 + \frac{4\eta^6}{s^4} \right] f^2 + 2 \left(\frac{s^2}{\eta^2} - 2 \right) f(1-f) + \frac{s^4}{\eta^4} (1-f)^2 \quad (-25)$$

$$G_2(\eta) = \left[\frac{1}{s^2} \left(\frac{4}{s^2} + \frac{12}{s^2} - 39 \right) \eta - \frac{2}{s^2} \left(\frac{4}{s^6} + \frac{12}{s^4} - \frac{7}{s^2} - 12 \right) \eta^3 - \frac{4}{s^4} \left(\frac{4}{s^2} - 3 \right) \eta^5 - \frac{16}{s^6} \eta^7 \right] f^2 + \frac{4\eta}{s^2} (1-2f) f - \frac{2}{\eta} \left[(1-2f) f + \frac{s^2}{\eta^2} (1-f)^2 \right] \quad (-26)$$

$$\begin{aligned}
G_3(\eta) = & \left[-24 - \frac{1}{s^2} \left(\frac{12}{s^4} + \frac{36}{s^2} - 6s \right) \eta^2 + \frac{2}{s^2} \left(\frac{4}{s^6} + \frac{12}{s^4} + \frac{1}{s^2} - 24 \right) \eta^4 \right. \\
& \left. - \frac{4}{s} \left(\frac{4}{s^2} - 1 \right) \eta^6 + \frac{8}{s^6} \eta^8 \right] f^2 + 2 \left(1 - \frac{2\eta^2}{s^2} \right) (1 - 2f) f \\
& - 4 \left(\frac{s^2}{\eta^2} - 1 \right) (1 - f) f - \frac{2s^2}{2} \left(\frac{1}{\eta^2} - 1 \right) (1 - f)^2
\end{aligned} \quad (-27)$$

Note that these three functions are all finite on the axis $\eta = 0$ and that they all vanish at infinity.

To find the averaging distance λ_1 in a non-dimensional form appropriate to this problem, we evaluate the expression

$$\left(\frac{\lambda_1}{z} \right)^2 = a_1^2 = \frac{\int_0^\infty \Omega^4 r dr}{\int_0^\infty (\Omega')^2 r dr} = \frac{\int_0^\infty G_1^2(\eta) \eta d\eta}{\int_0^\infty [G_2^2(\eta) + G_3^2(\eta)] \eta d\eta} \quad (-28)$$

It should be observed that both integrals in this last equation are evaluated from $\eta = 0$ to $\eta = \infty$ over the surface $z = \text{constant}$, and that in this case the quantity z cancels identically from the right side of the equation. There is no integration with respect to z . The dimensionless averaging distance a_1 is constant and independent of z .

The integrals involved in Eq (-28) can easily be evaluated numerically, thus fixing a_1 . It is found that a_1 is roughly the same as s , the jet width parameter; this seems reasonable.

A similar procedure is involved in determining the integrals I^2 and J^2 and the resulting local length scale parameter. Also the effects of polar symmetry must be taken into account in the manner of Appendix B. Again the integrations are from $\eta = 0$ to $\eta = \infty$ over the surface $z = \text{constant}$, and again the variable cancels from the final result. Thus

$$I^2 = \frac{1}{z^8} \frac{\int_0^\infty G_1^2(\eta') e^{-\left(\frac{\eta' - \eta}{a_1}\right)^2} \sqrt{\eta'} \psi\left(\frac{\sqrt{\eta\eta'}}{a_1}\right) d\eta'}{\int_0^\infty e^{-\left(\frac{\eta' - \eta}{a_1}\right)^2} \sqrt{\eta'} \psi\left(\frac{\sqrt{\eta\eta'}}{a_1}\right) d\eta'} = \frac{I_1^2(\eta)}{z^8} \quad (-29)$$

$$J^2 = \frac{1}{z^{10}} \frac{\int_0^\infty [G_2^2(\eta') + G_3^2(\eta')] e^{-\left(\frac{\eta' - \eta}{a_1}\right)^2} \sqrt{\eta'} \psi\left(\frac{\sqrt{\eta\eta'}}{a_1}\right) d\eta'}{\int_0^\infty e^{-\left(\frac{\eta' - \eta}{a_1}\right)^2} \sqrt{\eta'} \psi\left(\frac{\sqrt{\eta\eta'}}{a_1}\right) d\eta'} = \frac{J_1^2(\eta)}{z^{10}} \quad (-30)$$

Hence the dimensionless local length scale becomes

$$\left(\frac{\lambda}{z}\right)^2 = a^2(\eta) = \frac{1}{z^2} \left(\frac{I^2}{J^2}\right) = \frac{I_1^2(\eta)}{J_1^2(\eta)} \quad (-31)$$

The integrals $I_1^2(\eta)$ and $J_1^2(\eta)$ must of course be evaluated numerically. It turns out that the local length scale $a(\eta)$ obtained in this way is a slowly varying function. The value remains nearly constant at about 0.1 across most of the jet, and declines only slowly at relatively large distances from the centerline.

The energy equation for the axi-symmetric turbulent jet may be written, first in cylindrical coordinates,

$$\begin{aligned} \left(\frac{\partial E}{\partial t}\right) = 0 = \epsilon \bar{\epsilon}^2 - \epsilon (2E)^{7/6} J^{1/3} - \frac{1}{r} \frac{\partial}{\partial r} [rUE]_z - \frac{\partial}{\partial z} [VE]_r \\ + \frac{1}{r} \frac{\partial}{\partial r} \left[r \gamma \epsilon \left(\frac{\partial E}{\partial r} \right) \right]_z + \frac{\partial}{\partial z} \left[\gamma \epsilon \left(\frac{\partial E}{\partial z} \right) \right]_r \end{aligned} \quad (-32)$$

The various functions which appear in the energy equation can now be represented as functions of the similarity coordinates z and r , as follows

$$\begin{aligned}
\epsilon &= \epsilon(\eta) & \Omega^2 &= \frac{\Omega_1^2(\eta)}{z^4} \\
E &= \frac{E_1(\eta)}{z^2} & J^2 &= \frac{J_1^2(\eta)}{z^5} \\
U &= \frac{U_1(\eta)}{z} & r &= z\eta \\
V &= \frac{V_1(\eta)}{z} & \frac{\partial}{\partial r}(\) &= \frac{1}{z} \frac{\partial}{\partial \eta}(\)
\end{aligned} \tag{-33}$$

Upon substituting the expressions(-33) into the energy equation, we obtain,

$$\begin{aligned}
\frac{\epsilon \Omega_1^2}{z^4} - \epsilon \frac{(2E_1)^{7/6} J_1^{1/3}}{z^4} - \frac{1}{z^4 \eta} \frac{\partial}{\partial \eta} [\eta U_1 E_1] - \frac{\partial}{\partial z} \left[\frac{V_1 F_1}{z^3} \right]_r \\
+ \frac{1}{z^4 \eta} \frac{\partial}{\partial \eta} \left[\eta \gamma \epsilon \left(\frac{\partial F_1}{\partial \eta} \right) \right] + \frac{\partial}{\partial z} \left[\gamma \epsilon \frac{\partial}{\partial z} \left(\frac{F_1}{z^2} \right) \right]_r = 0
\end{aligned} \tag{-34}$$

This result confirms the fact that all terms of the energy equation are proportional to z^{-4} and that the above energy law is indeed compatible with the self-similarity of the overall solution. In order to expand this so as to be able to eliminate z from the result, we make use of the relation

$$\frac{\partial}{\partial z} \left[\]_r = \frac{\partial}{\partial z} \left[\]_\eta - \frac{\eta}{z} \frac{\partial}{\partial \eta} \left[\]_z \tag{-35}$$

In this way we obtain the general result

$$\begin{aligned}
\epsilon \Omega_1^2 - \beta (2E_1)^{7/6} J_1^{1/3} - \frac{1}{\eta} \frac{\partial}{\partial \eta} [\eta U_1 E_1] + 3V_1 E_1 + \eta \frac{\partial}{\partial \eta} [V_1 F_1] \\
+ \frac{1}{\eta} \frac{\partial}{\partial \eta} \left[\eta \gamma \epsilon \left(\frac{\partial F_1}{\partial \eta} \right) \right] + 3\gamma \epsilon \left[2F + \eta \left(\frac{\partial F_1}{\partial \eta} \right) \right] + \eta \frac{\partial}{\partial \eta} \left[\gamma \epsilon \left(2E_1 + \eta \frac{\partial E_1}{\partial \eta} \right) \right] = 0
\end{aligned} \tag{-36}$$

Considerable simplification can be achieved with little loss of accuracy by neglecting the effect of the slight divergence of the jet on the energy equation. This amounts to the assumption that

$$\frac{\partial}{\partial z} \left[\right]_r \approx 0 \quad (-37)$$

In this case the energy equation simplifies to

$$\epsilon \Omega_1^2 - \beta (2E_1)^{7/6} J_1^{1/3} - \frac{1}{n} \frac{\partial}{\partial n} [n U_1 E_1] + \frac{1}{n} \frac{\partial}{\partial n} \left[n \gamma \epsilon \left(\frac{\partial E_1}{\partial n} \right) \right] = 0 \quad (-38)$$

To complete the solution it is necessary to know the empirical functions α , β , and γ as defined by Eqs. (-18), (-19), and (-20) of Section 1.3. In this case, however, there are no fixed walls involved, and these quantities become constants, namely,

$$\begin{aligned} \alpha &= 0.065 \\ \beta &= \frac{1}{3.7} \\ \gamma &= 1.4 \end{aligned} \quad (-39)$$

The eddy viscosity $\epsilon(\eta)$ which occurs in the energy equation may be found in either of two ways. Firstly, we may use the solution previously found from the momentum analysis, by numerical integration of Eq. (-15). Secondly, we may use the basic postulate of the heuristic model, that is,

$$\epsilon = \alpha \lambda \sqrt{2E} = \alpha a \sqrt{2E_1} \quad (-40)$$

This second alternative represents a somewhat more severe test of the heuristic model and was adopted for the numerical example here discussed. Thus substituting Eq. (-40) into (-38) gives

$$\alpha a \sqrt{2E_1} \Omega_1^2 - \beta (2E_1)^{7/6} J_1^{1/3} - \frac{1}{n} \frac{\partial}{\partial n} [n U_1 E_1] + \frac{1}{n} \frac{\partial}{\partial n} \left[n \gamma a \sqrt{2E_1} \left(\frac{\partial E_1}{\partial n} \right) \right] = 0 \quad (-41)$$

The energy function $E_1(\eta)$ is the only unknown remaining in this equation. The boundary conditions are

$$\text{At } \eta = 0 \quad \left(\frac{\partial E_1}{\partial \eta} \right) = 0 \quad (-42)$$

$$\text{At } \eta \rightarrow \infty \quad E_1 \rightarrow 0 \quad (-43)$$

Eq. (-41) must be solved numerically. It happens to be convenient for this purpose to use the substitution

$$2E_1 = q_1^2 \quad (-44)$$

The equation is then expanded and the derivatives replaced by appropriate finite difference approximations in a straightforward manner.

The boundary condition (-43) at infinity must be satisfied indirectly. Actually, the integration starts at the centerline using $\left(\frac{\partial q}{\partial \eta} \right)_0 = 0$ and a trial value of q_0 . A value of q_0 is found by iteration which yields $q \approx 0$ at some large value of η . In this case the value $\eta = 0.25$ may be regarded as "large." It is found that the precise position of the outer boundary makes virtually no difference in the results as long as it is greater than about 0.20.

The energy distribution found in this way is shown by the solid line in Fig. 1.6-2. The agreement with the experimental results of Corrsin, Ref. (7), Laurence, Ref. (8), is seen to be satisfactory, especially in view of the degree of scatter in the data points.

Once the energy distribution is known the eddy viscosity is fixed by Eq. (-40) and the corresponding shear stresses by Eq. (-20). These are shown by the solid line in Fig. 1.6-3. The dashed line shows the corresponding shear stresses as computed from the momentum equation. Data points are from Corrsin, Ref. (9).

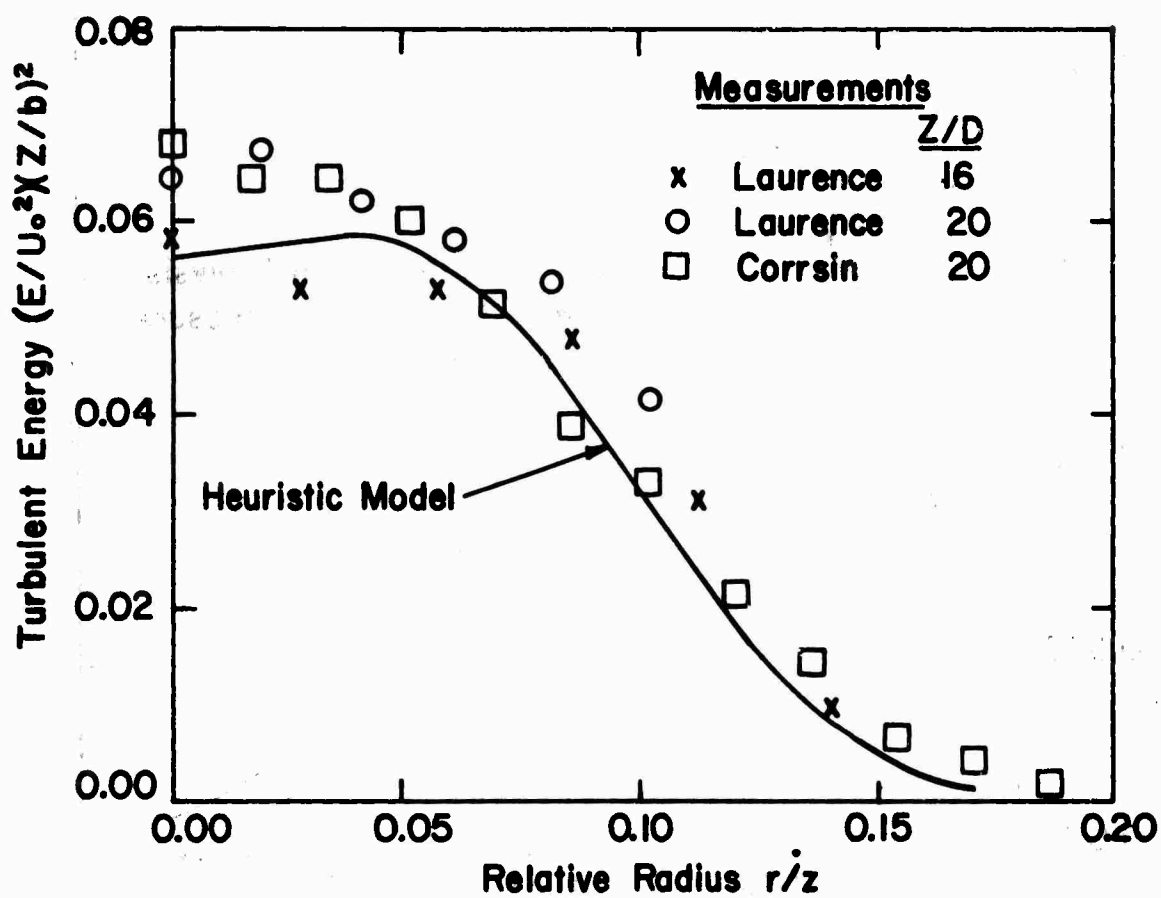


FIGURE I.6-2 TURBULENT ENERGY DISTRIBUTION IN A CIRCULAR JET

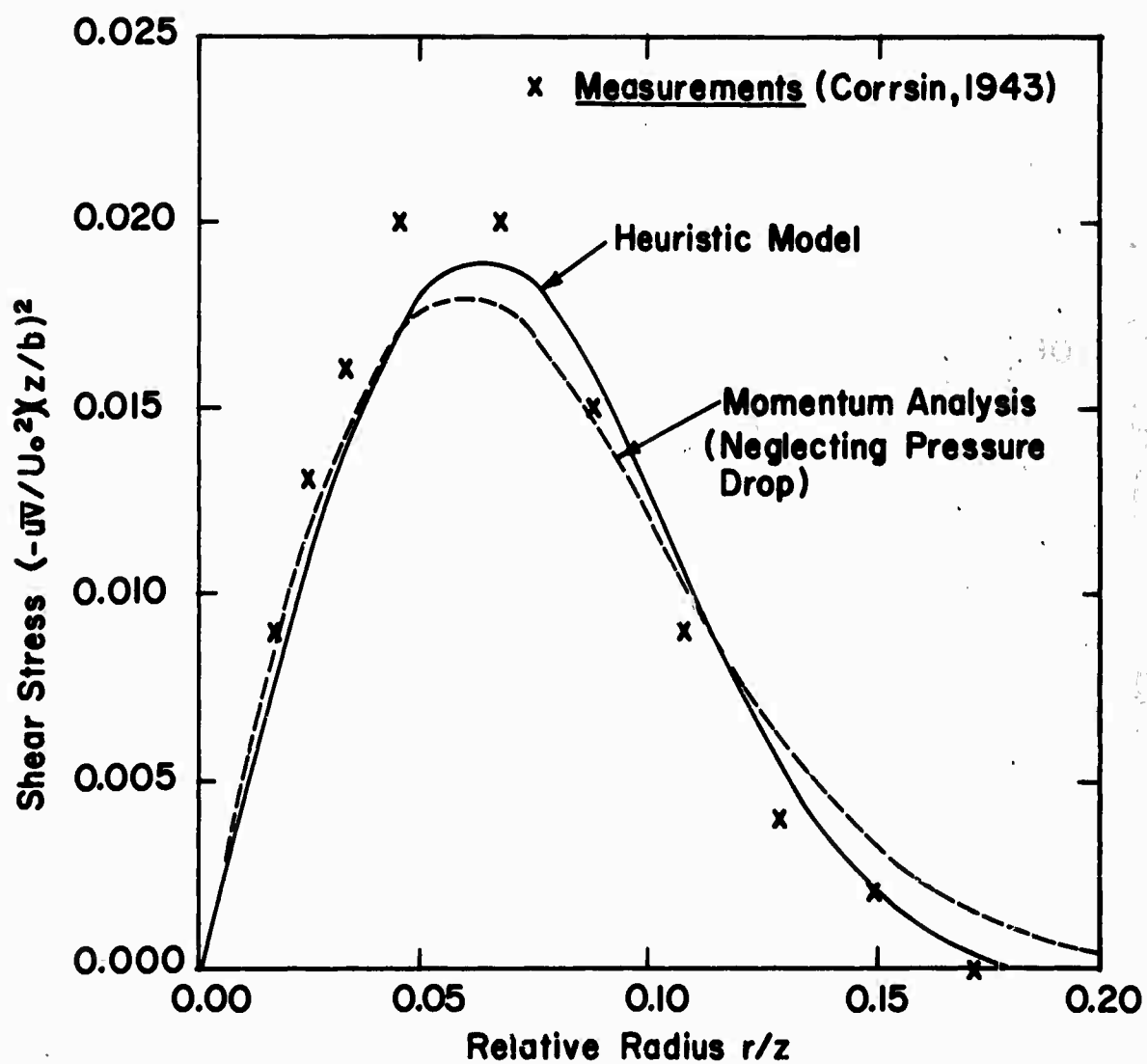


FIGURE I.6-3 REYNOLDS SHEAR STRESS DISTRIBUTION IN A CIRCULAR JET

If the heuristic model were completely correct, and if the assumed Gaussian velocity profile were exact, then the two shear stress curves shown in Fig. 1.6-2 would coincide exactly. As it is, the degree of agreement attained is considered to be reasonably satisfactory. The discrepancy between the two curves is smaller than the discrepancy between the experimental points and either of the curves.

It seems probable that in this instance the theoretically computed values are actually more accurate than the experimentally measured ones; the experimental measurement is innately difficult and uncertain.

It may be concluded that, on a whole, these results for the turbulent jet substantiate quite well the proposed heuristic model of fluid turbulence.

1.7 CONCLUSIONS AND RECOMMENDATIONS

It is concluded, on the basis of the evidence available so far, that the proposed heuristic model is basically adequate for determining the principle flow characteristics in the general case of inhomogeneous and non-stationary turbulence in incompressible flow.

It is recommended that the present model be applied also to other cases including pipe flow, two and three dimensional wakes, boundary layers and to some examples of unsteady flow. The aim would be to refine the present unified theory and extend its range of applicability.

It is also recommended that further experimental information be obtained in connection with those aspects of the model for which the present data are insufficient. These aspects include, for example,

- (a) the relation between the local length scale λ of the mean flow and the correlation length λ^* of the turbulence,
- (b) the generalized three dimensional stress/strain rate relations which actually exist in regions of strong anisotropy such as in the flow near a wall, and
- (c) the influence of various key parameters on the rate of dissipation of turbulent energy into heat.

Part II

Theory and Background of the Heuristic Model

by

T. H. Gawain, D.Sc.

2.1 THE FUNDAMENTAL PROBLEM

Consider a general turbulent flow field, one which is both inhomogeneous and non-stationary. How can the turbulent structure of such a field be adequately analyzed and described?

In principle, it is possible to formulate a valid general method for computing numerically the detailed solution $u_i^{(n)}(\mathbf{x}, t)$ for a particular realization of the field, subject to specified boundary conditions and initial conditions. Once these conditions are fully specified, the equations of continuity and motion suffice to establish a fully determinate and detailed solution. In theory, this solution procedure could be repeated very many times to generate a whole ensemble of such detailed realizations. All realizations would be compatible with the same general macroscopic boundary conditions and initial conditions. However, they would differ more or less randomly in regard to the fine scale details of the boundary and initial conditions, and therefore also in the fine scale details of the resulting motions.

Once such an ensemble of detailed solutions were available, any desired statistical features of the turbulence could then be computed from this ensemble in a completely determinate way.

It is significant that the foregoing method does, in principle, satisfy the mathematical requirements for a determinate solution. This solution is based solely and strictly on the general equations of continuity and motion (the Navier Stokes equations), and on the pertinent boundary and initial conditions for the particular case in question.

In practice, however, the volume of detailed calculations required to carry out the above hypothetical scheme of calculation for any case of practical interest is simply overwhelming. It lies vastly beyond the capabilities of any computer yet known or contemplated. The basic reason for this, of course, is that the detailed space/time structure of turbulence is extremely fine grained and enormously intricate. Hence for all practical purposes, the above hypothetical solution procedure must be ruled out.

There is a classical method for attempting to circumvent the above difficulty. Since it is impractical to solve the detailed equations and average the results, the classical method averages the basic equations themselves. It then attempts to compute the desired average characteristics of the flow from these averaged equations. Now, if the basic equations were linear, this technique would indeed solve the problem. Unfortunately, they are not. As a result of the non-linearity, the averaging process introduces additional unknowns, the Reynolds stresses, into the basic equations. Consequently, there are finally more unknowns than available equations. Therefore, the averaged equations do not comprise a closed and determinate set. This constitutes the so-called closure problem of turbulence theory.

It should be emphasized that the deficiency which arises from averaging the equations is fundamental. No amount of mere manipulation of the equations, however involved, can remedy this defect. The averaging of the equations leads to a definite loss of essential information in comparison with the information implicit in the original equations.

Therefore, so long as we are constrained to work only with the averaged equations instead of with the originals, it becomes unavoidably

necessary to invoke additional hypotheses in order to compensate for the lost information and to reestablish a determinate set of equations. Unfortunately, there is no known way to establish the required hypotheses deductively from the original fundamental equations themselves. Consequently, it is initially far from clear just what auxiliary hypotheses are most nearly adequate and satisfactory.

Obviously, the basic justification for any particular hypotheses finally adopted must therefore rest on a comparison of the consequences deduced from these hypotheses with experimental observations. This is the accepted scientific method of inductive inference. While it is to be regretted that the problem apparently cannot be solved on a wholly deductive basis, there are no a-priori reasons for despairing over the prospects for ultimate success by the inductive method.

It should also be pointed out at this juncture that the averaged equations of motion describe exactly the effect of the Reynolds stresses on the mean motion. However, they give no information whatever concerning the reciprocal effect of the mean motion on the Reynolds stresses. Yet some definite reciprocal effect of this kind unquestionably does exist. This, therefore, is precisely the problem with which our auxiliary hypotheses must necessarily deal.

2.2 STATISTICAL DESCRIPTION OF TURBULENCE

Consider an arbitrary and general flow field which is neither homogeneous in space nor stationary in time in regard to its turbulent structure. Let $u_1^{(n)}(\bar{x}, t)$ denote any one particular realization of this field. This means that the velocity components $u_1^{(n)}(\bar{x}, t)$ represent one

particular detailed solution of the equations of continuity and motion; this solution is compatible with the specified macroscopic boundary conditions and initial conditions. There exist, of course an infinity of such possible individual realizations which satisfy these general conditions. The separate realizations differ from one another only with respect to their fine scale details. We denote each realization by a distinct value of the index n , which is taken to be an integer. Thus we may have, for example, $n = 1, 2, 3, \dots, N$ where N is some indefinitely large number. Such a very large and representative collection of individual realizations we term an ensemble.

Fortunately, the particular details of any one individual realization are seldom of direct interest. What is usually required is statistical information concerning the principal features of the ensemble as a whole. In particular, with respect to any variable of interest, what we would generally like to know are not the individual values for particular realizations, but rather the following items of information. These are listed in order of decreasing urgency. Thus

- (a) The average value of the variable over the ensemble.
- (b) The statistical variance of the quantity with respect to the ensemble average.
- (c) The detailed probability distribution of the quantity over the ensemble.

For example, if $U_1^{(n)}(\bar{x}, t)$ is the instantaneous total velocity at space/time point \bar{x}, t for realization n , we require the corresponding ensemble mean

$$U_1(\bar{x}, t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N U_1^{(n)}(\bar{x}, t) \quad (2.2-1)$$

This result defines the mean flow field. Of course, the turbulent fluctuations of the n th realization are defined as the deviations from this mean. Thus

$$u_1^{(n)}(\bar{x}, t) = U_1^{(n)}(\bar{x}, t) - U_1(\bar{x}, t) \quad (-2)$$

From this definition, it follows that the turbulent fluctuation $u_1^{(n)}$ has zero mean over the ensemble.

The corresponding variance of this velocity fluctuation is given by

$$\overline{u_1^2(x, t)} = \lim_{N \rightarrow \infty} \frac{1}{(N-1)} \sum_{n=1}^N u_1^{(n)2}(\bar{x}, t) \quad (\text{No sum on index } i) \quad (-3)$$

Finally, the probability function which defines the detailed statistical distribution of this random variable u_1 over the ensemble may be denoted by the functional $\Psi[u_1^{(n)}(\bar{x}, t)]$. Here the quantity Ψdu_1 represents the probability that for a realization n , drawn at random from the ensemble, the velocity component u_1 at point \bar{x}, t will have a value which lies between u_1 and $(u_1 + du_1)$. If the space/time coordinates x, t be held constant at prescribed values, $\Psi(u_1)$ reduces to a corresponding ordinary probability density distribution. Hence the functional Ψ amounts to an ensemble of probability distributions, one for each space/time point of the field.

If the turbulence happens to be stationary with respect to time in its statistical features, then it becomes permissible and convenient to replace the ensemble average by a simple time average over a single realization. The realization index n may therefore be dropped. For example, the mean velocity distribution may now be redefined as follows.

$$U_1(\bar{x}) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} U_1'(\bar{x}, t) dt \quad (-4)$$

Similar changes apply to the definitions of variance and probability distribution.

Likewise, if the turbulence is homogeneous with respect to some line, surface or volume, the ensemble average may be replaced by the corresponding spatial average.

In the following analysis, the average will always be an ensemble average, unless specified otherwise. Also the discussion will center largely on certain important averages and variances, but will not deal specifically with the corresponding detailed probability distributions.

2.3 CORRELATION FUNCTIONS

In order to formulate an adequate theory of turbulence, it is necessary to define certain basic concepts which characterize the turbulence in the general vicinity of any prescribed space/time point. In this connection, the two point velocity correlation tensor is of basic importance.

Let P denote a reference space/time point with coordinates \bar{x}, t regarded as arbitrary but fixed. Let P' and P'' be two associated points with coordinates

$$\bar{x}' = \bar{x} + \Delta\bar{x}$$

$$t' = t + \Delta t$$

and

(2.3-1)

$$\bar{x}'' = \bar{x} - \Delta\bar{x}$$

$$t'' = t - \Delta t$$

The quantities $2\Delta\bar{x}$, $2\Delta t$ represent the space and time separation between the points P' and P'' and are regarded as variable. Note, however that

P' and P'' are always disposed symmetrically equal and opposite with respect to the fixed reference point P.

Components of the turbulent velocity fluctuations at the variable points P' and P'' are denoted by

$$u_1' = u_1(\bar{x} + \Delta\bar{x}, t + \Delta t) \quad \text{at } P'$$

and

$$u_1'' = u_1(\bar{x} - \Delta\bar{x}, t - \Delta t) \quad \text{at } P''$$

The mean product of these two velocity components is the tensor

$$\overline{u_1' u_1''} = \overline{u_1(\bar{x} + \Delta\bar{x}, t + \Delta t) u_1(\bar{x} - \Delta\bar{x}, t - \Delta t)} \quad (-3)$$

where the overbar denotes an ensemble average.

At zero separation, the above tensor reduces (upon reversal of algebraic sign) to the ordinary Reynolds kinematic stress at the reference point P. Thus

$$\tau_{ij} = -\overline{u_i u_j} = -\overline{u_k(\bar{x}, t) u_j(\bar{x}, t)} \quad (-4)$$

By reason of its symmetry the stress tensor $\overline{u_i u_j} \equiv \overline{u_j u_i}$ has at most six distinct components. The general velocity correlation tensor $\overline{u_i' u_j''}$, however, is not necessarily symmetrical with respect to an interchange of the indices and may therefore have up to nine distinct components.

Upon contracting the Reynolds stress tensor, we obtain the physically important quantity

$$q^2(\bar{x}, t) - \overline{u_i u_i} = \overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2} = 2E(\bar{x}, t) \quad (-5)$$

Here E represents the ensemble mean turbulent energy at the point \bar{x}, t and q^2 is the corresponding mean square velocity.

We now define the space/time correlation tensor R_{ij} merely by normalizing $\overline{u_i' u_j''}$ with respect to the value of q^2 at the fixed reference point P. Thus

$$R_{ij} = R_{ij}(\bar{x}, t; \Delta \bar{x}, \Delta t) = \frac{\overline{u_i u_j}}{q^2} = \frac{\overline{u_i(\bar{x} + \Delta \bar{x}, t + \Delta t) u_j(\bar{x} - \Delta \bar{x}, t - \Delta t)}}{q^2(\bar{x}, t)} \quad (-6)$$

It is of interest to compare the above definition which we have adopted with certain alternative forms which are sometimes encountered. One such alternative is

$$R'_{ij} = \frac{\overline{u_i(\bar{x}, t) u_j(\bar{x} + \Delta \bar{x}, t + \Delta t)}}{q(\bar{x}, t) q(\bar{x} + \Delta \bar{x}, t + \Delta t)} \quad (-7)$$

Another alternative is

$$R_{ij} = \frac{\overline{u_i(\bar{x}, t) u_j(\bar{x} + \Delta \bar{x}, t + \Delta t)}}{\sqrt{\overline{u_i^2(\bar{x}, t)}} \sqrt{\overline{u_j^2(\bar{x} + \Delta \bar{x}, t + \Delta t)}}} \quad (\text{No sum on } i \text{ or } j) \quad (-8)$$

The form we have chosen to prefer because it is related in a simpler fashion to the energy spectrum of turbulence, and because it has certain useful symmetry properties which will be pointed out presently.

Two important special cases of the foregoing are the spatial and temporal correlation tensors, defined respectively as follows:

$$R_{ij}^{(s)} = R_{ij}(\bar{x}, t; \Delta \bar{x}, 0) = \frac{\overline{u_i(\bar{x} + \Delta \bar{x}, t) u_j(\bar{x} - \Delta \bar{x}, t)}}{q^2(\bar{x}, t)} \quad (-9)$$

$$R_{ij}^{(T)} = R_{ij}(\bar{x}, t; 0, \Delta t) = \frac{\overline{u_i(\bar{x}, t + \Delta t) u_j(\bar{x}, t - \Delta t)}}{q^2(\bar{x}, t)}$$

Of particular significance are the first invariants of these two tensors, defined as follows:

$$R_{11}^{(s)} = R_{11}(\bar{x}, t; \Delta \bar{x}, 0) = \frac{u_1(\bar{x} + \Delta \bar{x}, t) u_1(\bar{x} - \Delta \bar{x}, t)}{q^2(\bar{x}, t)} \quad (-10)$$

$$R_{11}^{(T)} = R_{11}(\bar{x}, t; 0, \Delta t) = \frac{u_1(\bar{x}, t + \Delta t) u_1(\bar{x}, t - \Delta t)}{q^2(\bar{x}, t)}$$

Of course, the summation convention applies as usual to the repeated indices in the foregoing expression. Note that $R_{11}^{(s)}$ is an even function with respect to the independent variable Δx . Likewise $R_{11}^{(T)}$ is even with respect to Δt . These important symmetry properties are a consequence of defining R_{ij} according to Eq. (-6), rather than according to the alternative (-7) or (-8).

Because of their fundamental character and intimate relation to the turbulent energy, the scalar correlation functions $R_{11}^{(s)}$ and $R_{11}^{(T)}$ provide the natural bases for establishing appropriate length and time scales of the turbulence in the neighborhood of the space/time point \bar{x}, t .

Specifically, let $dv' = dx_1' dx_2' dx_3'$ represent an infinitesimal element of volume, and dt' an infinitesimal element of time at the variable point P' . Also let

$$\Delta \bar{x} \cdot \Delta \bar{x} = \Delta x_1 \Delta x_1 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2$$

represent the square of the spatial separation PP' .

Then the desired characteristic length λ^* and time τ^* may be defined as follows:

$$\lambda^{*2} = \lambda^{*2}(\bar{x}, t) = \frac{\int |R_{11}^{(s)}| (\Delta \bar{x} \cdot \Delta \bar{x}) dv'}{\int |R_{11}^{(s)}| dv'} \quad (-11)$$

$$\tau^{*2} = \tau^{*2}(\bar{x}, t) = \frac{\int |R_{11}^{(T)}| (\Delta t)^2 dt'}{\int |R_{11}^{(T)}| dt'} \quad (-12)$$

The integrals in (-11) extend over all space at a fixed time t . The integrals in (-12) extend over all time at a fixed spatial point x . Of course, these integrals are finite, despite the infinite domains of integration, because of the rapid decay in the correlation functions at large separations.

Note that the integrands contain only the absolute values of the correlation functions; this ensures that every non-zero element of the correlation, even if locally negative in some places, makes a positive contribution to the integrals which define λ^{*2} and τ^{*2} .

The numerators of Eqs. (-11) and (-12) represent the second moments of the respective correlation functions. They are therefore analogous to moments of inertia. Also, the quantities λ^{*2} and τ^{*2} are analogous to variances in probability theory. It is well known that such second moments attain their minimum values when evaluated with respect to the centroidal point of the distribution in question. In the present instance the correlation functions are symmetrical, with their centroids at the respective origins of coordinates, namely at $\Delta \bar{x} = 0$ and at $\Delta t = 0$. Hence the characteristic quantities λ^{*2} and τ^{*2} represent the minimum possible "variances" of the respective correlation functions. These properties make λ^* and τ^* particularly appropriate measures of the length and time scales of turbulence in the vicinity of the arbitrary space/time point \bar{x}, t .

2.4 SPECTRUM OF TURBULENCE

The space/time correlation tensor R_{ij} may be transformed into the wave number/frequency domain by means of multi-dimensional Fourier

integrals. Let $\kappa_1 \kappa_2 \kappa_3$ be the components of the spatial wave number vector $\vec{\kappa}$. Let ω be temporal wave number, that is, angular frequency. Also let $dv_{\kappa} = d\kappa_1 d\kappa_2 d\kappa_3$ represent an infinitesimal element of "volume" in wave number space. Note therefore that

$$\vec{\kappa} \cdot \Delta \vec{x} = \kappa_1 \Delta x_1 = \kappa_1 \Delta x_1 + \kappa_2 \Delta x_2 + \kappa_3 \Delta x_3 \quad (2.4-1)$$

We may now define the Fourier transform pair

$$\phi_{ij}(\vec{x}, t; \vec{\kappa}, \omega) = \frac{1}{\sqrt{2\pi}} \iint R_{ij}(\vec{x}, t; \Delta \vec{x}, \Delta t) e^{+i(\vec{\kappa} \cdot \Delta \vec{x} + \omega \Delta t)} dv' dt' \quad (-2)$$

$$R_{ij}(\vec{x}, t; \Delta \vec{x}, \Delta t) = \frac{1}{\sqrt{2\pi}} \iint \phi_{ij}(\vec{x}, t; \vec{\kappa}, \omega) e^{-i(\vec{\kappa} \cdot \Delta \vec{x} + \omega \Delta t)} dv_{\kappa} d\omega \quad (-3)$$

The integrations in Eq. (-2) extend over all of space/time $\Delta \vec{x}, \Delta t$ with parameters \vec{x}, t held constant. The integrations in (-3) extend over all of wave space $\vec{\kappa}, \omega$ with parameters \vec{x}, t again held constant.

The spectral correlation function ϕ_{ij} like its Fourier transform R_{ij} is a nine component tensor. Naturally these two related functions exhibit corresponding symmetry properties.

The first invariant ϕ_{11} like its Fourier transform R_{11} is a scalar and has a corresponding basic significance. It defines the character of the energy spectrum in the vicinity of the space time point \vec{x}, t . Incidentally, it can be shown from energy considerations that ϕ_{11} must be non-negative everywhere.

It may be seen that the functions R_{ij} and ϕ_{ij} are exactly equivalent in the information they provide concerning the structure of the turbulent field. The reason is simply that if either one of these functions be specified, the other one can be found from the above Fourier integral relations.

It is clear therefore that fundamental scales of length and time, which were previously developed in terms of $R_{11}^{(s)}$ and $R_{11}^{(T)}$ can equally well be defined in terms of the corresponding transforms $\phi_{11}^{(s)}$ and $\phi_{11}^{(T)}$.

Because of the above equivalence between the correlation functions R_{1j} and the spectral functions ϕ_{1j} it is immaterial in terms of which of these our turbulence theory is developed. We have chosen to emphasize the correlation aspects, and it is seen that there is no loss of information involved in so doing.

It should be noted that the nine spectral functions $R_{1j}(\bar{x}, t; \Delta \bar{x}, \Delta t)$ are ostensibly eight dimensional, the independent variables being $x_1, x_2, x_3, t, \Delta x_1, \Delta x_2, \Delta x_3, \Delta t$. Similarly, the nine spectral functions $\phi_{1j}(\bar{x}, t; \bar{k}, \omega)$ are also eight dimensional, the independent variables being $x_1, x_2, x_3, t, k_1, k_2, k_3, \omega$.

In addition to the above eight independent dimensions which are explicit, there is an additional dimension which is implicit. Recall that the correlation functions are themselves averages theoretically computable from a large ensemble of individual flow realizations. We may assign to each realization a superscript index n which varies from one to some very large number. Then n is equivalent to an additional independent dimension. For if we wish to know not just the ensemble averages themselves, but also something about their statistical distributions, then it becomes necessary to take into account the possible variations of the independent realization n from the ensemble mean.

On the other hand, any specialization of the conditions reduces the number of independent variables which must be considered. Thus, for stationary turbulence, R_{1j} becomes independent of t (but not of Δt) For

completely homogeneous turbulence, R_{ij} becomes independent of x_1, x_2, x_3 . Also, for isotropic conditions, spherical symmetry prevails, and R_{ij} ceases to depend on $\Delta x_1, \Delta x_2, \Delta x_3$ individually, but becomes a function only of their resultant $\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$. Furthermore, in this case the number of distinct components of R_{ij} reduces from nine to two, depending only on whether the indices i, j are equal or unequal. These are vast simplifications. They account for the fact that homogeneous and isotropic turbulence has been studied so much more extensively than any other case.

It should be noted that the fundamental phenomena of turbulence as represented by the ensemble of functions $u^{(n)}(\bar{x}, t)$ have five independent variables, namely, x_1, x_2, x_3, t and n , and three dependent variables, namely, u_1, u_2 , and u_3 . It is a curious fact that when we seek to make these phenomena more comprehensible by means of the correlation functions R_{ij} , the number of dependent and independent variables, instead of being sharply reduced, are actually increased! Of course, the functions R_{ij} have a regular and symmetrical structure, they vary smoothly, and die away rapidly at large separations, whereas the u_i vary in an extremely complex and erratic way. It is this qualitative difference that makes the correlation functions seem so much more comprehensible.

2.5 RELATION BETWEEN TURBULENT CORRELATION SCALES AND MEAN FLOW

It is customary and in many ways convenient to resolve fluid turbulence into two distinct complementary aspects which we then refer to as "the mean flow" and "the turbulent fluctuations", respectively. While

such a distinction is useful for certain conceptual and computational purposes, it should not be forgotten that these two aspects are inextricably coupled within a single unified process. In this connection it is pertinent to note that while we often find it convenient to introduce this distinction between mean flow and turbulent fluctuation into the basic Navier Stokes equations, the distinction is not innate within the original equations themselves. This is shown by the fact that these equations, being non-linear, cannot be fully decoupled into two independent sets, one dealing exclusively with the mean flow, the other dealing exclusively with the turbulent fluctuations.

The foregoing fact provides a useful hint in connection with the problem of formulating appropriate auxiliary hypotheses, as required for supplementing the averaged equations of motion. We have seen that any such hypotheses will naturally be formulated in terms of the fundamental physical parameters which characterize the overall turbulence in the vicinity of a given space/time point \bar{x}, t . Among these quantities are the mean square velocity $q^2(\bar{x}, t)$ and the macroscales of turbulence in space and time, as expressed by the previously defined parameters λ^* and τ^* .

Of course, in most problems of practical importance, interest centers chiefly on the mean flow field. One reason for this is that the mean flow is usually much more evident physically, much more convenient to observe and measure, seemingly more easy to understand. We tend to regard the turbulent fluctuations as a nuisance which has to be dealt with in order to get to the mean flow, and we tend to confine attention to those aspects of the turbulence whose connection to the mean flow is obvious and unavoidable.

More often than not, we do not have available the data which would be needed to establish such largely unseen characteristics of the turbulence as the spatial macroscale λ^* and temporal macroscale τ^* . Therefore, it is of doubtful utility to formulate any heuristic theory of turbulence directly in terms of these quantities. For example, what good does it do to express the Reynolds stresses, say, in terms of λ^* and τ^* if the latter quantities are initially just as much unknown as the Reynolds stresses themselves? Consequently, we must at this stage of our knowledge replace the fundamental macroscales λ^* and τ^* in our formulation by other quantities which are more or less equivalent, but which are far more accessible to observation and calculation.

Fortunately, for this purpose we can take advantage of the intimate connection as noted above, which necessarily exists between the local turbulence field and the local mean flow field. Presumably because of this connection, these local scales of space and time must somehow be reflected in the local characteristics of the mean flow, and must therefore be deducible from the mean flow.

While the exact nature of this relationship between the macroscales and the mean flow is far from clear we can legitimately infer that the relationship is inherently somewhat local in character. This follows from a fact universally revealed by all observed correlation curves. These always show a rapid decrease in correlation with increasing separation in space and time. Moreover, the correlations become negligible at separations larger than about two or three multiples of λ^* or τ^* .

Consequently, we infer, or at least we hypothesize, that quantities more or less equivalent to the true macro scales λ^* and τ^* can be defined

and computed in terms of the observable characteristics of the local mean flow. Let us call these quantities the apparent macroscales and designate them by the symbols λ and τ to distinguish them from the true macroscales λ^* and τ^* as earlier defined. The important problem of just how λ and τ should be defined in terms of the observable mean flow characteristics is taken up in a subsequent section. For the present we observe simply that if λ and τ be defined in some suitable manner, then it seems probable from the above reasoning that the dimensionless ratios

$$\frac{\lambda}{\lambda^*} = f_{\lambda} \quad (2.5-1)$$

$$\frac{\tau}{\tau^*} = f_{\tau} \quad (-2)$$

will be very stable and well behaved universal functions which vary but slowly if at all. We assume that for many practical applications it will turn out to be premissible to approximate these universal functions by suitable average values, and to treat them as simple quasi-constants. The actual numerical values of these ratios is regarded as a matter to be determined empirically. Of course, these numerical values will depend on the exact definitions that are finally adopted for λ and τ . However, the presumption is that if the definitions adopted are near optimal, the ratios will turn out to be roughly of order unity.

It follows from the foregoing discussion that our definitions of the apparent macroscales $\lambda(\bar{x}, t)$, $\tau(\bar{x}, t)$ in the vicinity of an arbitrary space/time point \bar{x}, t should depend only on mean flow conditions in a finite region surrounding that point. This principle is more realistic than any method which seeks to define these macroscales in terms of mean flow quantities only at the point \bar{x}, t itself. On the other hand, it avoids the opposite extreme which would hold that $\lambda(\bar{x}, t)$ and $\tau(\bar{x}, t)$ are somehow

directly dependent on conditions at all points in the flow field, however remote in space or time. Our proposed model accords more nearly than either of these alternatives with the observed fact that turbulent correlations are effectively limited to certain finite separations.

Naturally, the definitions of $\lambda(\bar{x},t)$ and $\tau(\bar{x},t)$ should emphasize mean flow conditions in the immediate vicinity of the point \bar{x},t , should give progressively less weight to conditions farther away, and should finally neglect mean flow effects at points sufficiently remote in space or time from point \bar{x},t . This requirement suggests the general idea of defining λ and τ in terms of appropriate weighted averages of certain pertinent mean flow quantities.

The preferred choice of weighting functions for this purpose would be the correlation functions $R_{ii}^{(s)}$ and $R_{ii}^{(T)}$ themselves, if these were known. Of course, if these functions were known, they could be used to find the true macroscales λ^* and τ^* directly, and there would then be no need for weighting functions. However, they are not known. Consequently, we are obliged to use heuristic substitutes, preferably functions which resemble the correlation functions to a certain extent. Fortunately, great accuracy is not necessary in this regard, for weighted averages tend to be relatively insensitive to minor variations in the form of the weighting functions used.

In view of these considerations, we choose the Gaussian curve as an appropriate form of heuristic weighting function. It expresses the general trends of correlation in a suitable manner, and has various convenient and known mathematical properties as well. Consequently, we finally write the space/time weighting function as follows

$$W = e^{-3 \frac{\Delta \vec{x} \cdot \Delta \vec{x}}{\lambda^2} - \frac{(\Delta t)^2}{\tau^2}} \quad (-3)$$

The reason for inserting the numerical factor 3 in the first term of the exponent will be explained presently.

Of course, for purely spatial weighting, we may delete the temporal term in the exponent and set

$$W^{(s)} = e^{-3 \frac{\Delta \vec{x} \cdot \Delta \vec{x}}{\lambda^2}} \quad (-4)$$

Similarly, for purely temporal weighting, we may delete the spatial term in the exponent and set

$$W^{(T)} = e^{-\frac{(\Delta t)^2}{\tau^2}} \quad (-5)$$

The above weighting functions $W^{(s)}$ and $W^{(T)}$ may be regarded as initial heuristic approximations to the true correlation functions $R_{ii}^{(s)}$ and $R_{ii}^{(T)}$, respectively. Note that all four of these functions equal unity at zero separation.

If we now replace the true correlation functions by their heuristic Gaussian approximations in Eqs. (2.3-11) and (2.3-12), we obtain the interesting results

$$\lambda^{*2} = \lambda^2 \quad (-6)$$

$$\tau^{*2} = \tau^2 \quad (-7)$$

This shows that for Gaussian correlation functions with exponential constants λ and τ , the true macroscales λ^* and τ^* may be simply identified with these exponential constants. Of course, this simple identification is possible only if the numerical factor 3 is inserted into the spatial exponent as indicated. This arises from the fact that the spatial integration of Eq. (2.3-11) is three dimensional, whereas the temporal integration of Eq. (2.3-12) is only one dimensional.

Recall, however, that the actual correlation functions are not necessarily Gaussian and are in fact unknowns in the present analysis. Moreover, for the reasons previously given, the quantities λ and τ will be defined not in terms of the correlation functions, but heuristically in terms of the mean flow. In fact the basic reason for introducing the above weighting functions in the first place is that they are needed in connection with these heuristic definitions of λ and τ . However, the weighting functions in turn contain λ and τ as parameters. It is therefore apparent that the actual calculation of λ and τ will necessarily involve some type of iteration process.

For certain purposes in the subsequent analysis it will prove convenient to utilize a normalized form of the above Gaussian weighting function. In this connection, consider the following integrals, namely,

$$H = \iint W dv' dt' \quad (-8)$$

$$H^{(s)} = \iint W^{(s)} dv' dt' \quad (-9)$$

and

$$H^{(T)} = \iint W^{(T)} dv' dt' \quad (-10)$$

As usual, the integrands extend over the entire space/time domain. Of course, the integrals themselves are always finite even if the domain of integration extends to infinity.

We now define the normalized weighting functions as follows:

$$w = \frac{W}{H} \quad (-11)$$

$$w^{(s)} = \frac{W^{(s)}}{H^{(s)}} \quad (-12)$$

$$w^{(T)} = \frac{W^{(T)}}{H^{(T)}} \quad (-13)$$

When normalized in this way, the above weighting functions have the following convenient properties, that is,

$$\iint w dv' dt' = 1 \quad (-14)$$

$$\iint w^{(s)} dv' dt' = 1 \quad (-15)$$

$$\iint w^{(T)} dv' dt' = 1 \quad (-16)$$

2.6 LOCAL LENGTH AND TIME SCALES OF THE MEAN FLOW

We have been considering any arbitrary and general turbulent flow field of which the mean flow may be either steady or unsteady. With every space/time point of this field may be associated two parameters λ and τ which express respectively a length scale and a time scale characteristic of the mean flow pattern in the vicinity of the point. Our problem is to devise suitable explicit definitions of these parameters in a way which meets various essential requirements. Some of these have already been discussed. There are some additional requirements as well, among which are the following:

- (1) The parameters λ and τ must be true scalars, and therefore invariant with respect to any rotation, reflection, translation or acceleration of the reference axes.
- (2) These parameters should also be everywhere continuous, finite and positive (except possibly in certain limited singular regions, such as at a solid boundary).
- (3) If feasible, the mean flow quantities λ^2 and τ^2 should preferably be related to the correlation quantities λ^{*2} and τ^{*2} , respectively, such that the ratios $\frac{\lambda^2}{\lambda^{*2}}$ and $\frac{\tau^2}{\tau^{*2}}$ are close to unity, or at least so that these ratios remain as nearly constant as possible.

These requirements can be satisfied as follows. Let

$$U_i = U_i(x_1, x_2, x_3, t) = U_i(\bar{x}, t) \quad (2.6-1)$$

represent the velocity field of the mean flow. Then the corresponding strain rate tensor is

$$\Gamma_{ij} = \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (-2)$$

Of course, we are considering here only the incompressible case, for which the first invariant vanishes, that is

$$\Gamma_{ii} = 2 \left(\frac{\partial U_i}{\partial x_i} \right) = 2 \left[\left(\frac{\partial U_1}{\partial x_1} \right) + \left(\frac{\partial U_2}{\partial x_2} \right) + \left(\frac{\partial U_3}{\partial x_3} \right) \right] = 0 \quad (-3)$$

Consequently, there is no net dilatation, and the Γ_{ij} represent purely distortional effects.

We now define a generalized strain rate Ω , and a generalized strain rate gradient Ω' by means of the following expressions, namely

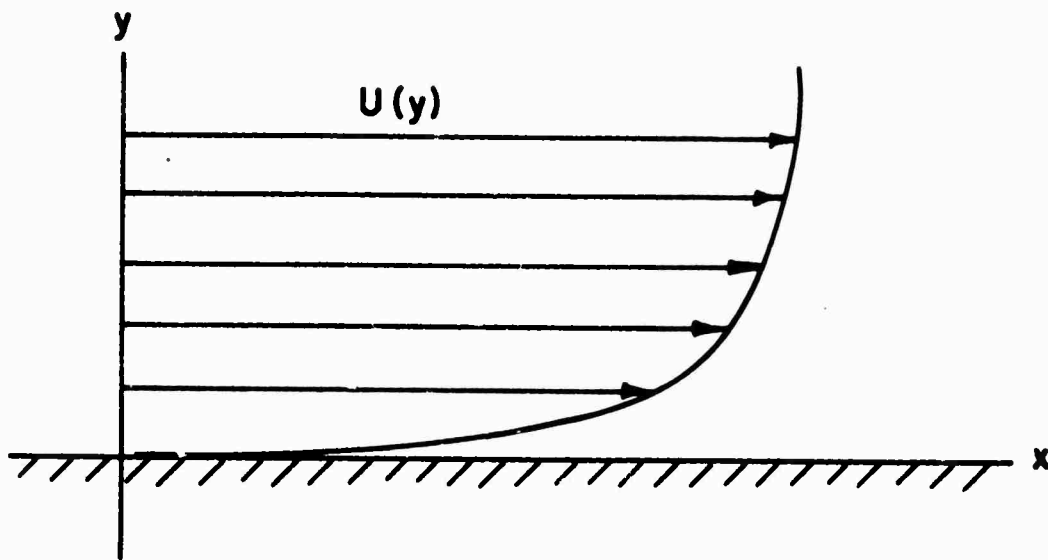
$$\Omega^2 = \frac{1}{2} \Gamma_{ij} \Gamma_{ij} \quad (-4)$$

$$\Omega'^2 = \left(\frac{\partial \Omega}{\partial x_i} \right) \left(\frac{\partial \Omega}{\partial x_i} \right) \quad (-5)$$

From these definitions, the following relation can be obtained. It will prove useful in the subsequent work. Thus

$$(\Omega \Omega')^2 = \frac{1}{4} \left(\frac{\partial \Omega^2}{\partial x_i} \right) \left(\frac{\partial \Omega^2}{\partial x_i} \right) \quad (-6)$$

The factor 1/2 which appears in Eq. (-4) calls for a brief explanation. Consider a simple plane parallel shear flow such as illustrated in Fig. 2.6-1. Let the velocity be parallel to axis x_1 , and let the shear gradient be in the direction of x_2 . Then $\Gamma_{11} = \Gamma_{22} = \Gamma_{33} = \Gamma_{23} = \Gamma_{31} = 0$. Now the only non-zero shear strain rate is $\Gamma_{12} = \Gamma_{21}$. Then applying Eq. (-4)



In this case the definitions of Ω and Ω' reduce to

$$\Omega = U = (dU/dy)$$

$$\Omega' = U'' = (d^2U/dy^2)$$

FIGURE 2.6-1 EXAMPLE : SHEAR QUANTITIES Ω AND Ω' IN PLANE PARALLEL FLOW

we find that, because of the presence of the factor 1/2, we obtain the simple result

$$\Omega = \Gamma_{12}$$

In other words the presence of the optional factor 1/2 identifies Ω with the ordinary shearing rate in a simple plane shear flow. Eq. (-4) may be regarded as an extension of this concept to a general state of strain rate.

It can be shown that while U_i and Γ_{ij} depend on the orientation of the reference axes, the quantity Ω is invariant in this regard, and is therefore a true scalar. The same is true of the quantities Ω^1 , Ω^2 , Ω'^2 and $(\Omega\Omega')^2$. Hence λ^2 and τ^2 can be conveniently defined in terms of any independent pair of these quantities.

To illustrate this idea, consider the expressions

$$\lambda^2 = C_\lambda^2 \frac{\Omega^2}{\Omega'^2} \quad (-8)$$

$$\tau^2 = C_\tau^2 \frac{1}{\Omega^2} \quad (-9)$$

where C_λ^2 and C_τ^2 are dimensionless constants. The combinations shown may be seen to define local scales of length and time, λ and τ . In fact, Eq. (-8) will be recognized as the expression for the von Karman mixing length, or rather, as a generalization of the concept, for von Karman's original definition was restricted to the special case of parallel flow.

One difficulty with the above approach, however, is that the functions λ^2 and τ^2 so defined tend to fluctuate too wildly instead of being smooth and well behaved. Moreover, at certain singular points where Ω^2 or Ω'^2 are locally zero, the quantities λ^2 or τ^2 may reach values of zero or infinity, neither of which is premissible or physically meaningful.

The cure for this difficulty, of course, is to replace the local point values of Ω^2 and Ω'^2 by suitable space/time averages evaluated in the vicinity of the point. Thus we may write

$$\Omega^2 = \iint w \Omega^2 dv' dt' \quad (-10)$$

$$\Omega'^2 = \iint w \Omega'^2 dv' dt' \quad (-11)$$

whereupon

$$\lambda^2 = C_\lambda^2 \frac{\overline{\Omega^2}}{\Omega'^2} \quad (-12)$$

$$\tau^2 = C_\tau^2 \frac{1}{\Omega'^2} \quad (-13)$$

The new values of λ^2 and τ^2 obtained in this way will now vary smoothly as required, and will never be either zero or infinite. The dimensionless constants C_λ^2 and C_τ^2 remain to be specified.

Note that the normalized form of the weighting function is used in the integrals (-10) and (-11). This is important because it ensures that the integrals remain relatively insensitive to the precise form of the weighting function.

For the reasons given earlier, a Gaussian curve has been chosen for the weighting function. For consistency, the characteristic exponential constants λ and τ of this function should be respectively equal to the λ and τ computed from Eqs. (-12) and (-13). In other words, the averaging intervals in length and time should be approximately equal to the characteristic local scales of length and time. Some iteration will be involved in satisfying this requirement.

It turns out to be useful to generalize the above idea even further. For this purpose we define the following integrals, namely,

$$I^2 = \iint w(\Omega^2)^a (\Omega'^2)^b dv' dt' \quad (-14)$$

$$J^2 = \iint w(\Omega^2)^c (\Omega'^2)^d dv' dt' \quad (-15)$$

where the exponents a, b, c, d may be assigned arbitrarily. However, for reasons which will soon be evident, we stipulate that

$$D = \begin{vmatrix} b & d \\ (a+b) & (c+d) \end{vmatrix} \neq 0 \quad (-16)$$

Moreover, to avoid improper integrals we require all of the above exponents to be non-negative.

Now, depending only on how the arbitrary exponents a, b, c, d are specified, the integrals I^2 and J^2 may be assigned various physical dimensions. Consequently, it is always possible to define a mean square length λ^2 and mean square time τ^2 by expressions of the following general form, namely,

$$\lambda^2 = C_\lambda^2 (I^2)^k (J^2)^l \quad (-17)$$

$$\tau^2 = C_\tau^2 (I^2)^m (J^2)^n \quad (-18)$$

where C_λ^2 and C_τ^2 are arbitrary dimensionless constants. It is easily shown on dimensional grounds that the four exponents k, l, m, n are given by the following formulas:

$$\begin{aligned} k &= -\frac{1}{D}(c+d) & m &= +\frac{d}{D} \\ l &= +\frac{1}{D}(a+b) & n &= -\frac{b}{D} \end{aligned} \quad (-19)$$

It is now seen that Eq. (-16) represents a necessary condition to ensure the existence of the above solution.

The foregoing formulas show that, subject only to the condition (-16), specification of the six arbitrary constants a , b , c , d , C_λ^2 and C_τ^2 uniquely defines the quantities λ^2 and τ^2 . These six quantities are here regarded as empirical constants whose values can in principle be chosen so as to confer upon the defined quantities λ^2 and τ^2 a wide range of desired properties.

Ideally, it would be desirable to choose these six constants in such a way as to make λ^2 and τ^2 agree as nearly as possible with λ^{*2} and τ^{*2} , respectively, over a broad range of conditions. In principle, this can always be done if and when sufficient experimental data becomes available for this purpose. At the present time, the available data are insufficient for such an analysis. It therefore becomes necessary, at least for the time being, to resort to expedients of a more heuristic nature, as will be explained presently.

The calculation of the space/time integrals I^2 and J^2 which fix λ^2 and τ^2 is seen to be inherently lengthy and complicated. One reason is that the space/time integrals involved are, in their present form, four dimensional. The second reason is that these same quantities appear not only as the final unknowns, but also as the exponential constants in the Gaussian weighting function. Hence our next endeavor is to simplify the analysis as much as possible in both these respects.

In this connection, we observe first that in computing the integrals I^2 and J^2 , if the mean flow happens to be steady, the integration over time cancels identically from the result, and the integrals reduce to ordinary volume integrals. Secondly, we notice that if the flow field also

happens to be either plane or axis-symmetric, all volume integrals can be reduced to corresponding surface integrals. Also if the flow field happens to be of the type known as self-similar, all volume integrals again reduce to surface integrals. If the flow field is both axis-symmetric and self-similar, then a double reduction is possible, and the volume integrals can be reduced to line integrals. These are tremendous simplifications.

By a self-similar flow field, we mean a field in which there exists a family of surfaces across which the mean velocity distribution has a common pattern; the velocity distribution across the various surfaces differs, at most, only in the respective scales of length and time, or what amounts to the same thing, of length and velocity. Many common flows in jets, wakes, ducts and boundary layers are known to be self-similar. The characteristic surfaces in such cases are conveniently taken as cross-sections approximately normal to the principal flow direction.

This consideration of self-similarity shows us, thirdly, that for such flows each characteristic surface possesses its own overall length and time scales; these are constant over any one such surface, but may differ from one surface to the next. In addition to these constant overall scales of length and time which characterize the cross-section as a whole, there are also distinct local scales of length and time, as considered earlier, in the vicinity of any reference point on the surface. These local scales, unlike the overall scales, vary smoothly from point to point along the cross-sectional surface.

Of course, even when we have taken these factors into account to simplify our formulas, the definitions of λ^2 and τ^2 will remain indeterminate until the four exponents a , b , c , d and the two proportionality

constants C_λ^2 and C_τ^2 are definitely specified. Obviously, there exists an enormous amount of flexibility here, and this was done intentionally. The flexibility has been deliberately provided in order to permit matching λ^2 and τ^2 as closely as possible to λ^{*2} and τ^{*2} over a broad range of conditions.

Naturally, to do this properly will require much data beyond what is now available. Hence at the present time our scheme has far more flexibility than we really need or can properly exploit.

Consequently, we have found it necessary to simplify at this time in what may seem to be a somewhat arbitrary manner. Actually, such simplifications were guided by a fair amount of numerical experimentation. According to present evidence, they seem to give reasonably satisfactory agreement with physical experiment. No doubt, there remains room for further improvement, however.

Specifically, for our present turbulence model, we have chosen the following numerical values for the four arbitrary exponents, namely,

$$a = 2 \quad b = 0 \quad c = 1 \quad d = 1 \quad (-20)$$

Also, it proves to be most convenient to choose the proportionality factor simply as

$$C_\lambda^2 = 1 \quad (-21)$$

The other proportionality factor C_τ^2 is not required for the present simplified analysis.

Let us summarize the situation as it now stands after the foregoing simplifications. The integrals below extend over the entire flow field. Thus

$$\Delta \bar{x} = (\bar{x}' - \bar{x}) = \text{separation variable} \quad (-25)$$

$$W(\vec{x}, \vec{x}') = e^{-\frac{\Delta \vec{x} \cdot \Delta \vec{x}}{\lambda^2}} = \text{weighting function} \quad (-26)$$

$$H(\vec{x}) = \int W dv' = \text{normalizing integral} \quad (-27)$$

$$w(\vec{x}, \vec{x}') = \frac{1}{H} W = \text{normalized weighting function} \quad (-28)$$

$$I^2(\vec{x}) = \int w \Omega^4 dv' = \text{first characteristic integral} \quad (-29)$$

$$J^2(\vec{x}) = \int w (\Omega \Omega')^2 dv' = \text{second characteristic integral} \quad (-30)$$

$$\lambda^2(\vec{x}) = \frac{I^2}{J^2} = \text{local length scale parameter} \quad (-31)$$

Notice that Eq. (-26) above differs from Eq. (2.5-4) in that the unessential factor 3 has been eliminated from the exponent for convenience. This is arbitrary but permissible; it amounts to a rescaling of λ .

The foregoing relations suffice in principle to define the required scale factor λ at every point in the flow field as required, but do involve a computational difficulty. The problem is that the value of λ is needed for Eq. (-26) before a computed value is available from Eq. (-31). Hence, it is necessary to resort to an iteration process such as the following. Let the $(n+1)$ th approximation to λ be defined as:

$$\lambda_{n+1}^2(\vec{x}) = I_{n+1}^2(\vec{x}) / J_{n+1}^2(\vec{x}) \quad (-32)$$

where

$$I_{n+1}^2(\vec{x}) = \int_{\text{all space}} W_n(\vec{x}, \vec{x}') \Omega^4(\vec{x}') dv' \quad (-33)$$

$$J_{n+1}^2(\vec{x}) = \int_{\text{all space}} W_n(\vec{x}, \vec{x}') [\Omega \Omega'(\vec{x}')]^2 dv' \quad (-34)$$

The weighting function W_n is defined as follows. Let

$$W_n(\vec{x}, \vec{x}') = \exp - \frac{(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{\lambda_n^2(\vec{x})} \quad (-35)$$

Then

$$w_n(\vec{x}, \vec{x}) = \frac{W_n(\vec{x}, \vec{x})}{\int_{\text{all space}} W_n(\vec{x}, \vec{x}') dv'} \quad (-36)$$

We now define (in principle)

$$\lambda^2(\vec{x}) = \lim_{n \rightarrow \infty} \lambda_n^2(\vec{x}) \quad (-37)$$

Advantage should be taken, of course, of any knowledge concerning the flow to hasten the convergence of the λ_n 's. In particular, if the flow is self-similar, such as the flow in a pipe, duct, or boundary layer, or the far downstream region of a turbulent jet or wake, calculations need only be performed on a single representative cross-section of the flow, and the results rescaled according to the appropriate nondimensionalizing parameters.

As an initial guess to start the iteration process, we take $\lambda_0 = \infty$ everywhere. Then in computing λ_1 , we find that the weighting function W_0 is equal to unity everywhere, and so we obtain simply

$$\lambda_1^2 = \frac{\int_{\text{all space}} \Omega^4 dv}{\int_{\text{all space}} (\Omega \Omega')^2 dv} \quad (-38)$$

a constant independent of position. Thus, λ_2 is the first nonconstant approximation to λ that we obtain.

Numerical experiments with various flow fields have shown that, in general, the convergence of the λ_n 's is extremely rapid. This is illustrated in Table 2.6-1 which shows successive numerical approximations to the λ distribution in laminar flow in a two dimensional channel. Consequently, for the purposes of the calculations presented in this paper, the approximation was made:

$$\lambda(\vec{x}) \approx \lambda_2(\vec{x}) \quad (-39)$$

Some improvement could possibly be gained by continuing to higher approximations, but in view of the gross simplifications already inherent in the model as a whole, the authors feel that such refinement is not warranted. Moreover, absolute accuracy in this regard is not essential. It is necessary, however, that the chosen order of approximation be adhered to consistently. It is also important to adjust the initially undetermined numerical constants of the model so that the chosen order of approximation gives satisfactory agreement with experimental data. The significant numerical constants are those that occur in the heuristic expressions for the quantities α , β , and γ .

TABLE 2.6-1

Convergence of Macroscale Distribution in a Two-Dimensional Channel
with Parabolic (Laminar) Velocity Profile^a

y/b	λ_n/b				
	<u>n = 0</u>	<u>n = 1</u>	<u>n = 2</u>	<u>n = 3</u>	<u>n = 20</u>
0.0	∞	0.775	0.695	0.674	0.664
0.2	∞	0.775	0.706	0.693	0.689
0.4	∞	0.775	0.731	0.727	0.727
0.6	∞	0.775	0.758	0.758	0.758
0.8	∞	0.775	0.782	0.781	0.781
1.0	∞	0.775	0.802	0.800	0.800

^a 2b = channel width y = distance from centerline.

2.7 REYNOLDS STRESSES AND GENERALIZED EDDY VISCOSITY CONCEPT

In dealing with an incompressible fluid, it is customary to divide the equations of motion through by the fluid density. As a result, the viscosity, pressure and stress terms all appear in these equations divided by the density. It is convenient to avoid writing out the density explicitly each time. Therefore we define the quotient of viscosity divided by density as the kinematic viscosity, and assign it the symbol ν . Similarly, the quotient of pressure divided by density becomes kinematic pressure, symbol p . Stress divided by density becomes kinematic stress, symbol τ_{ij} . Note that kinematic pressure and kinematic stress have dimensions of velocity squared.

The kinematic Reynolds stress tensor may now be written

$$\tau_{ij} = \tau_{ij} = -\overline{u_i u_j} \quad (2.7-1)$$

The contraction obtained by equating indices and summing over the repeated index defines a scalar invariant, namely,

$$\tau_{ij} = \tau_{11} + \tau_{22} + \tau_{33} = -(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}) = -q^2 = -2E \quad (-2)$$

Now the Reynolds stress tensor may be rewritten as the sum of a purely dilatational tensor plus a purely distortional tensor, that is,

$$\tau_{ij} = -\frac{1}{3}q^2 \delta_{ij} + \tau'_{ij} \quad (-3)$$

Conversely, the distortional part is defined by

$$\tau'_{ij} = \tau_{ij} + \frac{1}{3}q^2 \delta_{ij} \quad (-4)$$

Of course, the individual components of τ'_{ij} depend on the orientation of the coordinate axes. However, the double summation

$$\frac{1}{2}\tau'_{ij}\tau'_{ij} = \tau'^2 \quad (-5)$$

is invariant with respect to any change in the orientation of the reference axes at any point, and is therefore a true scalar. We may think of τ' as representing a kind of generalized scalar amplitude of the tensor τ'_{ij} .

Of course, the optional factor 1/2 was inserted into Eq. (-5) to maintain consistency with the corresponding formula for the strain rate invariant Ω . The physical justification is as follows. Consider a simple state of pure shear stress such that $\tau'_{11} = \tau'_{22} = \tau'_{23} = \tau'_{31} = 0$. Let the simple shear $\tau'_{12} = \tau'_{21}$ be the only non-zero stress. In this case Eq. (-5) gives

$$\tau' = \tau'_{12} \quad (-6)$$

Hence the invariant τ' may be identified with the magnitude of the simple shear stress itself. Eq. (-5) represents an extension of this concept to the general state of stress.

A somewhat similar analysis may be made of the rate of strain tensor of the mean flow field. Recall the definition

$$\Gamma_{ij} = \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (-7)$$

For an incompressible fluid, the associated dilational tensor vanishes, that is,

$$\Gamma_{11} + \Gamma_{22} + \Gamma_{33} = 2 \left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} \right) = 0 \quad (-8)$$

Consequently the tensor Γ_{ij} is itself wholly distortional. An amplitude scalar Ω may be defined through the invariant relation

$$\frac{1}{2} \Gamma_{ij} \Gamma_{ij} = \Omega^2 \quad (-9)$$

Next we wish to formulate an appropriate way to express the general relation that may exist between the distortional stress tensor τ'_{ij} and the distortional strain rate tensor Γ_{ij} . In this connection, it is instructive to consider as a preliminary model, the relation which is known to exist between the ensemble averaged distortional viscous stress $(\overline{\tau'_{ij}})_v$ and the distortional strain rates Γ_{ij} of the mean flow. This is given by

$$(\overline{\tau'_{ij}})_v = \nu \Gamma_{ij} \quad (-10)$$

where ν is the ordinary molecular kinematic viscosity.

This model suggests that we attempt to express the distortional Reynolds stresses in the form

$$\tau'_{ij} = \epsilon \Gamma_{ij} \quad (-11)$$

where ν is now replaced by ϵ , the so-called eddy kinematic viscosity.

However, this form of equation implies that

$$\frac{\tau'_{11}}{\Gamma_{11}} = \frac{\tau'_{22}}{\Gamma_{22}} = \frac{\tau'_{33}}{\Gamma_{33}} = \frac{\tau'_{23}}{\Gamma_{23}} = \frac{\tau'_{31}}{\Gamma_{31}} = \frac{\tau'_{12}}{\Gamma_{12}} \quad (-12)$$

Unfortunately, there is no a-priori reason to believe that the tensors τ'_{ij} and Γ_{ij} will necessarily satisfy the five similarity constraints expressed by Eqs. (-12). We therefore require a more general relationship, one which remains applicable even if τ'_{ij} and Γ_{ij} are both specified arbitrarily.

One method which is sometimes suggested for dealing with this requirement is to postulate a relation of the form

$$\tau'_{ij} = \epsilon_{ijkl} \Gamma_{kl} \quad (-13)$$

Now the eddy viscosity becomes a fourth order tensor. Since τ'_{ij} and Γ_{ij} each have six distinct components, the tensor ϵ_{ijkl} has thirty-six

components! By employing tensor arguments, the number of independent eddy viscosity components can actually be reduced to ten. Furthermore, if the tensors τ'_{ij} and Γ_{ij} are arbitrarily specified, Eqs. (-13) can in fact be reduced to an equivalent set of five independent equations in ten unknowns. However, these equations do not constitute a determinate set. Clearly, this type of formulation is both inadequate and overly complex as well.

A somewhat simpler tensor relation has been suggested by Hirt, Ref. (4). In our notation it becomes

$$\tau'_{ij} = \frac{1}{2}[\epsilon_{ik}\Gamma_{kj} + \epsilon_{jk}\Gamma_{ki}] \quad (-14)$$

While this is an improvement over Eq. (-13), it is not free of difficulties. If ϵ_{kl} is taken as an unsymmetrical tensor, Eq. (-14) fails to produce a determinate solution for ϵ_{kl} when τ'_{ij} and Γ_{ij} are specified. On the other hand, if ϵ_{kl} is taken as symmetrical, the relation (-14) loses generality; there are then certain theoretical combinations of τ'_{ij} and Γ_{ij} which are not reconcilable with (-14) no matter how the ϵ_{jk} are chosen.

The foregoing difficulties can be overcome by adopting the following relation between τ'_{ij} and Γ_{ij} , namely,

$$\tau'_{ij} = \epsilon[\Gamma_{ij} + \Omega f_{ij}] \quad (-15)$$

It will be shown that the quantities ϵ and f_{ij} suffice to define the relation between distortional stress and strain rate in a fully general way. If τ'_{ij} and Γ_{ij} are specified arbitrarily, a corresponding unique solution can be found for ϵ and f_{ij} . Conversely, if ϵ , f_{ij} , and Γ_{ij} are specified, then τ'_{ij} is uniquely determined.

Solving Eq. (-15) for f_{ij} gives

$$f_{ij} = \frac{1}{\Omega}[\frac{1}{\epsilon}\tau'_{ij} - \Gamma_{ij}] \quad (-16)$$

It is immediately apparent from this that

$$f_{ij} = f_{ji} \quad (-17)$$

and

$$f_{ii} = f_{11} + f_{22} + f_{33} = 0 \quad (-18)$$

However, as it stands, Eq. (-16) does not uniquely fix f_{ij} because it contains the as yet undetermined parameter ϵ . It will now be shown that for any arbitrarily prescribed τ'_{ij} and Γ_{ij} , a corresponding "preferred" value of ϵ can be uniquely defined. For this purpose let

$$\tau''_{ij} = \epsilon \Omega f_{ij} = \tau'_{ij} - \epsilon \Gamma_{ij} \quad (-19)$$

Here τ''_{ij} represents the discrepancy between τ'_{ij} and $\epsilon \Gamma_{ij}$. We wish to choose ϵ so as to minimize this discrepancy in some appropriate sense.

Hence we form the invariant of the discrepancy namely,

$$\begin{aligned} \tau''_{ij} \tau''_{ij} &= 2\tau''^2 = [\tau'_{ij} - \epsilon \Gamma_{ij}][\tau'_{ij} - \epsilon \Gamma_{ij}] \\ &= \tau'_{ij} \tau'_{ij} - 2\epsilon \Gamma_{ij} \tau'_{ij} + \epsilon^2 \Gamma_{ij} \Gamma_{ij} \end{aligned} \quad (-20)$$

or

$$\tau''^2 = \tau'^2 - \epsilon \Gamma_{ij} \tau'_{ij} + \epsilon^2 \Omega^2$$

Now to minimize the invariant τ''^2 of the discrepancy, we set

$$\frac{\partial \tau''^2}{\partial \epsilon} = 0 - \Gamma_{ij} \tau'_{ij} + 2\epsilon \Omega^2 = 0 \quad (-21)$$

whereupon

$$\epsilon = \frac{\Gamma_{ij} \tau'_{ij}}{2\Omega^2} = \frac{\Gamma_{ij} \tau'_{ij}}{\Gamma_{ij} \Gamma_{ij}} = \frac{\Gamma_{ij} \tau'_{ij}}{\Gamma_{ij} \Gamma_{ij}} \quad (-22)$$

This simple result is important. It is the basic definition of the scalar eddy viscosity ϵ for the case of general stress and strain rate.

Now multiplying (-19) by Γ_{ij} and using (-22) we obtain the significant consequence that

$$\Gamma_{ij} f_{ij} = 0 \quad (-23)$$

This is the mathematical expression of the concept that the tensors Γ_{ij} and f_{ij} are "orthogonal". Hence our basic stress/strain rate law, Eq. (-15), expresses the idea that the Reynolds stress tensor τ'_{ij} consists of a tensor $\epsilon \Gamma_{ij}$ which is simply proportional to Γ_{ij} , plus a tensor $\epsilon \Omega f_{ij}$ which is orthogonal to Γ_{ij} .

The tensor f_{ij} has nine possible components. The symmetry constraint (-17) reduces this to six distinct components. The non-divergence constraint (-18) reduces it further to five independent components. Finally, the orthogonality constraint (-23) reduces the degrees of freedom of f_{ij} to just four. Hence ϵ and f_{ij} together possess five degrees of freedom. This is precisely the necessary and sufficient number to express the general relation between distortional stress and mean flow strain rate. If we add $\frac{q^2}{3}$ as a sixth degree of freedom, this then allows us to define the general relation between overall Reynolds stress and mean flow strain rate.

Since the foregoing equations are pure tensor relations, they remain valid for all possible orientations of the coordinate axes. It is particularly convenient to choose the principal axes of strain rate in order to bring out the essential concepts in the simplest possible form. We use asterisks to designate these axes. Also, there is no loss in generality in ordering the subscripts so that $\Gamma_{11}^* \geq \Gamma_{22}^* \geq \Gamma_{33}^*$. Of course $\Gamma_{23}^* = \Gamma_{31}^* = \Gamma_{12}^* = 0$. Moreover, it happens to be useful to introduce the following auxiliary terminology. Let

$$\begin{aligned}\Gamma_{11}^* &= \frac{2}{\sqrt{3}} \Omega \sin\left(\frac{2\pi}{3} + \theta^*\right) \\ \Gamma_{22}^* &= \frac{2}{\sqrt{3}} \Omega \sin(0 + \theta^*) \\ \Gamma_{33}^* &= \frac{2}{\sqrt{3}} \Omega \sin\left(-\frac{2\pi}{3} + \theta^*\right)\end{aligned}\tag{-24}$$

The merit of Eqs. (-24) lies in the fact that they express the three quantities Γ_{11}^* , Γ_{22}^* , Γ_{33}^* in terms of just two parameters, namely, Ω and θ^* . Furthermore, as the reader may easily verify, eqs. (-24) satisfy identically the following important constraints,

$$\begin{aligned}\Gamma_{11}^* + \Gamma_{22}^* + \Gamma_{33}^* &= 0 \\ \frac{1}{2}(\Gamma_{11}^{*2} + \Gamma_{22}^{*2} + \Gamma_{33}^{*2}) &= \Omega^2\end{aligned}\tag{-25}$$

The parameter θ^* is then fixed from the relation

$$\frac{\sqrt{3} \Gamma_{22}^*}{(\Gamma_{11}^* - \Gamma_{33}^*)} = \tan \theta^*\tag{-26}$$

It is equally useful to adopt a similar scheme for the f_{ij}^* . Thus

$$\begin{aligned}f_{11}^* &= \frac{2}{\sqrt{3}} g^* \sin\left(\frac{2\pi}{3} + \psi^*\right) \\ f_{22}^* &= \frac{2}{\sqrt{3}} g^* \sin(0 + \psi^*) \\ f_{33}^* &= \frac{2}{\sqrt{3}} g^* \sin\left(-\frac{2\pi}{3} + \psi^*\right)\end{aligned}\tag{-27}$$

whereupon

$$\begin{aligned}f_{11}^* + f_{22}^* + f_{33}^* &= 0 \\ \frac{1}{2}(f_{11}^{*2} + f_{22}^{*2} + f_{33}^{*2}) &= g^{*2}\end{aligned}\tag{-28}$$

and

$$\frac{\sqrt{3} f_{22}^*}{(f_{11}^* - f_{33}^*)} = \tan \psi^*\tag{-29}$$

Now the orthogonality constraint, eq. (-23), reduces in this case to the form

$$\Gamma_{11}^* f_{11}^* + \Gamma_{22}^* f_{22}^* + \Gamma_{33}^* f_{33}^* = 0 \quad (-30)$$

Upon substituting Eqs. (-24) and (-27) into (-30) and reducing, we obtain the beautifully simple result

$$\psi^* = \theta^* \pm \frac{\pi}{2} \quad (-31)$$

This too expresses a kind of orthogonality between the Γ_{ij} and the f_{ij} .

With ψ^* now known, the components f_{11}^* , f_{22}^* , f_{33}^* are all fixed as soon as g^* is specified. The off-diagonal terms f_{23}^* , f_{31}^* , and f_{12}^* may be specified individually. The final stress/strain rate equations for these particular axes now taken the form

$$\begin{aligned} \tau_{11}^* &= -\overline{u_1^2} = -\frac{1}{3} q^2 + \varepsilon [\Gamma_{11}^* + \Omega f_{11}^*] \\ \tau_{22}^* &= -\overline{u_2^2} = -\frac{1}{3} q^2 + \varepsilon [\Gamma_{22}^* + \Omega f_{22}^*] \\ \tau_{33}^* &= -\overline{u_3^2} = -\frac{1}{3} q^2 + \varepsilon [\Gamma_{33}^* + \Omega f_{33}^*] \\ \tau_{23}^* &= -\overline{u_2 u_3} = 0 + \varepsilon [0 + \Omega f_{23}^*] \\ \tau_{31}^* &= -\overline{u_3 u_1} = 0 + \varepsilon [0 + \Omega f_{31}^*] \\ \tau_{12}^* &= -\overline{u_1 u_2} = 0 + \varepsilon [0 + \Omega f_{12}^*] \end{aligned} \quad (-32)$$

Note that the principal axes of stress and strain will coincide if, and only if, $f_{23}^* = f_{31}^* = f_{12}^* = 0$.

The further reduction of these equations for the important special case of a plane mean flow is interesting. Now the axes x_1 and x_3 are in the plane of the flow, and x_2 is normal to this plane. Hence we have

$$\begin{aligned} \theta^* &= 0 \\ \Gamma_{11}^* &= +G \end{aligned} \quad (-33)$$

$$\begin{aligned}\Gamma_{22}^* &= 0 \\ \Gamma_{33}^* &= -\epsilon.\end{aligned}\tag{-33}$$

Consequently from Eq. (-31) we find that

$$\psi^* = \pm \frac{\pi}{2}\tag{-34}$$

There is some choice concerning the manner of handling algebraic signs. We choose to drop the negative sign in (-34) and instead, to allow g^* in the following expressions to take on positive or negative values as required. Then

$$\begin{aligned}f_{11}^* &= -\frac{1}{\sqrt{3}} g^* & f_{23}^* &= 0 \\ f_{22}^* &= +\frac{2}{\sqrt{3}} g^* & f_{31}^* &= \text{arbitrary} \\ f_{33}^* &= -\frac{1}{\sqrt{3}} g^* & f_{12}^* &= 0\end{aligned}\tag{-35}$$

Of course, $f_{23}^* = f_{12}^* = 0$ by reason of symmetry. Hence g^* and f_{31}^* remain as the two arbitrary degrees of freedom which define the coefficients f_{ij}^* .

The final stress/strain rate equations become

$$\begin{aligned}\tau_{11}^* &= -\overline{u_1^2} = -\frac{1}{3} q^2 + \epsilon \Omega \left[1 - \frac{1}{3} g^* \right] \\ \tau_{22}^* &= -\overline{u_2^2} = -\frac{1}{3} q^2 + \epsilon \Omega \left[0 + \frac{2}{3} g^* \right] \\ \tau_{33}^* &= -\overline{u_3^2} = -\frac{1}{3} q^2 + \epsilon \Omega \left[-1 - \frac{1}{3} g^* \right] \\ \tau_{31}^* &= -\overline{u_3 u_1} = 0 + \epsilon \Omega f_{31}^* \\ \tau_{23}^* &= \tau_{31}^* = 0\end{aligned}\tag{-36}$$

Hence the overall stress/strain rate relation is now fully specified by the four quantities $\frac{1}{3} q^2$, ϵ , g^* , and f_{31}^* .

In the case of axi-symmetric mean flows, the results are similar to those for plane flow, except that θ^* is not in general zero; therefore ψ^* must be found from Eq. (-31) and the f_{ij}^* from Eq. (-27). However, we still have $f_{23}^* = f_{31}^* = 0$ for this case.

The basic stress/strain parameters are now $\frac{1}{3} q^2$, ϵ , g^* , f_{31}^* , and ψ^* . Note from Eqs. (-26) and (-31) that ψ^* is a known function of the strain tensor, but that the other four parameters may be arbitrary.

For purposes of approximation in connection with our heuristic turbulence model, we have decided to make the simplifying assumption $f_{ij} = 0$. Actually this is not as drastic as it seems. In the first place, as we have seen, f_{ij} embodies at most only four degrees of freedom, not six or nine. Secondly, we are mainly concerned with plane or axisymmetric mean flows, for this f_{ij} has only two degrees of freedom, namely, g^* and f_{31}^* . Thus this assumption amounts to neglecting but two degrees of freedom.

Secondly, an analysis of the momentum transfer involved in generating the Reynolds stresses suggests that the term $\epsilon \Gamma_{ij}$ accounts for most of the effect, and that the deviation $\epsilon \Omega f_{ij}$ should indeed be small. While the experimental data which is available is not decisive on this point, it does tend to support the above hypothesis. It does seem that this simplification may become somewhat inadequate in the immediate vicinity of a fixed wall, but it appears to be reasonably satisfactory elsewhere in the flow.

Under these assumptions, the principal axes of stress and strain coincide, and the equations for these particular axes reduce to the simple form

$$\begin{aligned} \tau_{11}^* &= -\frac{1}{3} q^2 + \epsilon \Omega \\ \tau_{22}^* &= -\frac{1}{3} q^2 + 0 \\ \tau_{33}^* &= -\frac{1}{3} q^2 - \epsilon \Omega \\ \tau_{23}^* &= \tau_{31}^* = \tau_{12}^* = 0 \end{aligned} \quad (-37)$$

The foregoing stress/strain rate law, despite its apparent simplicity, can be useful for calculating the Reynolds stresses which correspond to any specified mean flow strain rates only if the value of ϵ can somehow be estimated with acceptable accuracy. Whether this can be done is an empirical question, capable of being answered by suitable experimental information. The experimental data so far available is not yet adequate to answer this need fully. Further experiment and analysis is needed. Tentatively, however, we are justified in postulating that ϵ is some function of the characteristic local turbulence parameters q , λ , and τ as previously defined, perhaps also of molecular kinematic viscosity ν , distance y from the nearest wall, and possibly of some as yet unknown additional parameters as well. This can be written symbolically in the form

$$\epsilon = \epsilon(q, \lambda, \tau, \nu, y, \dots) \quad (-38)$$

Actually, we would be on slightly more fundamental ground to choose λ^* and τ^* in place of λ and τ . Unfortunately, λ^* and τ^* are not usually known, whereas λ and τ have been related in a specified manner to the mean flow. Consequently, for pragmatic reasons we replace λ^* and τ^* by λ and τ in the above expression.

The foregoing empirical law which governs the eddy viscosity ϵ can be greatly simplified by dimensional analysis. We recall that in mechanics generally, the number of fundamental dimensions is three. These are often taken as mass, length and time. However, in the case of incompressible flow, division of the equations of motion by density has the effect of eliminating mass from the problem. Our final equations are therefore all kinematical in character, with length and time as the only fundamental dimensions. It follows therefore that any two of the above parameters may be chosen as the necessary and sufficient basis for the non-dimensionalizing process.

From a purely logical standpoint, it would seem natural and convenient to use the basic local scales of length and time, λ and τ , for this purpose. However, there are grounds for believing that q and λ play roles which are physically more important than that played by τ . This seems to be especially true for mean flows which are steady, or which vary only slowly. Consequently, we choose q and λ and our non-dimensionalizing parameters.

As a result of the foregoing considerations, the empirical eddy viscosity law may now be rewritten in the dimensionless form.

$$\left(\frac{\epsilon}{q\lambda}\right) = \alpha \left(\frac{q\tau}{\lambda}, \frac{q\lambda}{v}, \frac{y}{\lambda}, \dots\right) \quad (-39)$$

Thus, our problem has been simplified to that of finding the dimensionless coefficient α which presumably varies as a function of the dimensionless parameters $\left(\frac{q\tau}{\lambda}\right)$ and $\left(\frac{q\lambda}{v}\right)$ and $\left(\frac{y}{\lambda}\right)$, and possibly also of one or two other dimensionless variables as well.

In practice, the usefulness of the above formulation hinges on whether the dimensionless coefficient α does in fact behave in a reasonably simple and stable manner. Our present indications are that this is indeed so. Owing to limitations, of time and data, we have not fully established the actual dependence, if any, of α on the quantities $\left(\frac{q\tau}{\lambda}\right)$ and $\left(\frac{q\lambda}{v}\right)$. We have four, heuristically, however, that satisfactory overall results can apparently be found by assuming that α depends only on the wall distance parameter (y/λ) . The following expression seems to give reasonably good curve fits, namely,

$$\alpha = 0.065 \left\{ 1 + e^{-\left(\frac{y}{\lambda} - 1\right)^2} \right\} \quad (-40)$$

Of course, this expression is tentative. More data is needed on this point in any case.

Naturally, this particular expression for α is applicable only so long as we specifically retain our present definition of λ . Any modification of the definition of λ clearly calls for a corresponding adjustment in α . Note however that the dimensionless quantity

$$\left(\frac{\epsilon}{\lambda^* q} \right) = \left(\frac{\alpha \lambda}{\lambda^*} \right) \quad (-41)$$

should remain substantially invariant with respect to such details.

However, even if α be accurately known, and λ as well, the actual eddy viscosity ϵ cannot be evaluated until q also is established. Bearing in mind that the turbulent energy is $E = \frac{1}{2} \sigma^2$, it is apparent that the determination of q must involve an energy analysis of the turbulence. Physically, this is fundamental. It ties together the turbulence, the eddy viscosity and the mean flow into a consistent overall picture. It also provides the additional equations which are necessary to form a determinate set.

2.8 HEURISTIC ENERGY ANALYSIS OF THE TURBULENT FIELD

Starting with the Navier Stokes equations of motion, it is possible to develop a corresponding exact energy equation, as shown in detail in Appendix A. The following result is obtained, namely,

$$\begin{aligned} \left(\frac{\partial E}{\partial t} \right) = & \overset{\text{I}}{\frac{1}{2} \tau_{ij} \Gamma_{ij}} - \overset{\text{II}}{\frac{\nu}{2} \gamma_{ij} \gamma_{ij}} - \overset{\text{III}}{\frac{\partial}{\partial x_i} [U_i E]} - \overset{\text{IV}}{\frac{\partial}{\partial x_i} \left[u_i \left(p + \frac{u_j u_j}{2} \right) \right]} \\ & \overset{\text{V}}{+ \frac{\partial}{\partial x_i} \left[\nu \left(\frac{\partial \Gamma_i}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} \right) \right]} \end{aligned} \quad (2.8-1)$$

This equation governs the evolution over time and distribution over space of the ensemble mean turbulent kinetic energy, that is, of

$$E = \frac{1}{2} \overline{q^2} = \frac{1}{2} \overline{u_j u_j} = \frac{1}{2} (\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}) \quad (-2)$$

The five terms on the right side of Eq. (-1) have the following physical significance:

Term I - The rate of energy input into turbulence as work done by the mean flow against the resistance of the Reynolds stresses.

Term II - The rate of irreversible energy dissipation into heat. Note that the quantity $\overline{\gamma_{ij} \gamma_{ij}}$ is positive definite. This quantity accounts for the local entropy production.

Term III - The rate of convective transport of turbulent energy E by the mean flow U_i .

Term IV - The rate of net transport by turbulent diffusion of the turbulent pressure plus kinetic energy fluctuations.

Term V - The rate of net transport of energy by molecular diffusion. Note that the Reynolds stresses, as well as the energy, appear in this term.

While Eq. (-1) is theoretically exact, it contains several terms which cannot be evaluated exactly without detailed knowledge of the turbulent fluctuations. Such detailed information is not practically attainable. It is therefore necessary to approximate these terms by suitable semi-empirical expressions. These expressions will necessarily contain empirical coefficients whose values can in principle be chosen so as to optimize agreement with available experimental data.

We have shown in the preceding section that the Reynold's stresses can always be expressed with complete exactness and generality in the form

$$-\overline{u_1 u_j} = \tau_{1j} = -\frac{1}{3} q^2 \delta_{1j} + \epsilon [\Gamma_{1j} + \Omega f_{1j}] \quad (-3)$$

where

$$\epsilon = \frac{\tau_{1j} \Gamma_{1j}}{\Gamma_{1j} \Gamma_{1j}} \quad (-4)$$

and

$$\Gamma_{1j} f_{1j} = 0 \quad (-5)$$

Consequently, term I, the work input expression in the energy equation reduces exactly to the result

$$\frac{1}{2} \tau_{1j} \Gamma_{1j} \equiv \epsilon \Omega^2 \quad (-6)$$

However, as we have seen earlier, ϵ must here be approximated by an expression of the form

$$\epsilon = \alpha \lambda q \quad (-7)$$

where α is presumably a well behaved empirical function, the expression for which has been given earlier.

Next consider the dissipation of energy into heat, as given by term II of the basic energy equation. Dimensional considerations suggest that this term can be expressed to good advantage in the form

$$\dot{E}_H = \frac{1}{2} \nu \overline{Y_{1j} Y_{1j}} = \frac{\nu q^2}{\lambda_D^2} \quad (-8)$$

This expression amounts to a definition of the so-called dissipation length λ_D which is a length characteristic of the microscale of turbulence. This form of expression would be very convenient if the dissipation length λ_D happened to be related to the macroscopic scale λ in some fairly simple and invariant way. Unfortunately, this does not seem to be the case.

However, experimental studies such of those by Laufer, Ref. (6), clearly suggest that at sufficiently high Reynolds numbers, the heat dissipation effects tend to become independent of Reynolds number! This implies that, contrary to Eq. (-8), the optimal form of expression for \dot{E}_H is one which does not contain viscosity ν explicitly. This will be the case in fact if we assume, for example, that

$$\left(\frac{\lambda}{\lambda_D}\right)^2 = \beta' \left(\frac{q\lambda}{\nu}\right) \quad (-9)$$

where β' is now hopefully some well behaved empirical function. For combining these last two results gives

$$\dot{E}_H = \beta' \left(\frac{q^3}{\lambda}\right) \quad (-10)$$

which is not explicitly dependent on ν . This form is similar to that suggested by Prandtl in Ref. (2).

The fact that the viscous dissipation occurs in a form which does not explicitly contain the viscosity seems rather paradoxical at first sight, and calls for some comment. It can be given a reasonable interpretation by considering turbulence energy processes from the spectral point of view. It is well known that the work input, term I, occurs mainly at the long wave length end of the spectrum, and that the dissipation to heat, term II, occurs mainly at the short wave length end. At high Reynolds numbers, there is a broad range of wavelengths in between, the so-called inertial range, for which both work input and heat dissipation are negligible. Each wave length within this inertial range simply receives energy from the longer wave length components and transmits this energy to the shorter wave length components. It would appear that the rate of transmission of energy thru the spectrum is largely controlled by this process in the

inertial range, rather than by the viscous dissipation process itself. Apparently the viscous dissipation rate easily adjusts itself as may be required to dissipate all of the energy coming through the inertial range. Since the viscous process is not rate controlling, the effect of viscosity tends to drop out of the experimental picture.

These experimental facts and theoretical considerations suggest that the heat dissipation should be analyzed by the same general method previously employed in connection with the eddy viscosity. For example, we might assume tentatively that

$$\dot{E}_H = \dot{E}_H(q, \lambda, \tau, \nu, y, \dots) \quad (-11)$$

Then non-dimensionalizing on the basis of q and λ as common parameters, we obtain

$$\left(\frac{\lambda \dot{E}_H}{q} \right) = \beta' = \beta' \left(\frac{q\tau}{\lambda}, \frac{\lambda q}{\nu}, \frac{y}{\lambda}, \dots \right) \quad (-12)$$

which is seen to be equivalent to Eq. (-10)

This result suggests that β' is a well behaved universal function which can be established empirically. However sound in principle, this idea is rather difficult to apply at the present time owing to the paucity of suitable experimental data. It does seem plausible that for mean flows which are not too strongly unsteady, the effect on β' of the parameter $\left(\frac{q\tau}{\lambda} \right)$ might prove to be negligible. On the other hand, efforts to establish β' as a function of $\left(\frac{\lambda q}{\nu} \right)$ alone have not as yet proved conclusive one way or the other. This is due in part to limitations in the data and in part to limitations in time.

Instead of following the above line of investigation to the end, we discovered the following more promising modification. Numerical experimentation suggested that it might provide a more satisfactory alternative.

Recall that the quantities λ and τ in our model are themselves computed from the integrals I and J. Hence it amounts merely to a rearrangement of form to rewrite Eq. (-11) as follows:

$$\dot{E}_H = \dot{E}_H(q, I, J, v, y, \dots) \quad (-13)$$

Now, non-dimensionalizing on the basis of q and J as common parameters, we obtain

$$\frac{\dot{E}}{(q^7 J)^{1/3}} = \beta \left[\frac{I}{(qJ)^{2/3}}, \frac{1}{v} \left(\frac{q^5}{J} \right)^{1/3}, \left(\frac{J}{q^2} \right)^{1/3} y, \dots \right] \quad (-14)$$

Actually, the significance of the dimensionless group on the left was established by certain somewhat speculative physical arguments which will not be repeated here. (See ref. 11) It was found, however, on the basis of numerical investigation, that this particular dissipation parameter β seems to remain more nearly constant, and to agree better with physical data than does the parameter β' resulting from the earlier analysis. Hence this is the form that was retained for our present heuristic turbulence model.

Here β is a slowly varying dimensionless function. The dependence of β on the quantities $\frac{I}{(qJ)^{2/3}}$ and $\frac{1}{v} \left(\frac{q^5}{J} \right)^{1/3}$ was not actually investigated, due to time limitations. However, it was found heuristically that β seems to depend only on the relative distance $\left(\frac{y}{\lambda} \right)$ to the nearest wall, if any, and that far from any walls β becomes essentially constant.

The following empirical equation seems to produce reasonably good curve fits, namely,

$$\beta = \frac{1}{3.7} \left\{ 1 + e^{-\left(\frac{y}{\lambda} - 1 \right)^2} \right\}^{-1} \quad (-15)$$

Some data which tends to confirm this formulation is given in Appendix C. In any case, it is clear that the energy dissipation to heat needs further investigation.

Next consider term IV of the basic energy equation. It is useful for purposes of discussion

$$\overline{u_1 \left(p + \frac{u_1 u_1}{2} \right)} = \overline{u_1 p} + \overline{u_1 \left(\frac{u_1 u_1}{2} \right)} \quad (-16)$$

The second of these terms clearly represents the net transport of the fluctuating turbulent kinetic energy $\left(\frac{u_1 u_1}{2} \right)$ by the action of the turbulent velocity fluctuations u_1 . Clearly the turbulent kinetic energy $\left(\frac{u_1 u_1}{2} \right)$ is both a scalar and an extensive property. It may be regarded as something which is physically transported along with the element of fluid mass. In this respect it is quite analogous to other extensive properties, such as salinity, for example. It is customary and appropriate to represent such net turbulent transport by an appropriate empirical diffusion coefficient. Hence we write

$$\overline{u_1 \left(\frac{u_1 u_1}{2} \right)} = -\epsilon' \frac{\partial}{\partial x_1} \left(\frac{u_1 u_1}{2} \right) = -\epsilon' \left(\frac{\partial E}{\partial x_1} \right) \quad (-17)$$

On the other hand the pressure velocity correlation term $\overline{u_1 p}$ is in a somewhat different category owing to the fact that pressure p is an intensive property. It is not an extensive property which can be regarded as being transported along with the fluid mass. We note however that p is a dependent variable. This means that whenever the spatial distribution of the velocity fluctuations u_1 is specified, the corresponding spatial distribution of p is likewise fixed. Hence, looking at the matter statistically, we can say that p is correlated in some fashion with u_1 . In fact,

taking into account dimensional considerations, we can improve this statement and say that p is statistically correlated with $\left(\frac{u_1 u_1}{2}\right)$. For example, in the limiting case of an inviscid fluid, Bernoulli's equation informs us that regions of higher than average kinetic energy will tend to be regions of lower than average pressure, and vice versa. Consequently we are justified in writing the pressure term in the analogous form

$$\overline{u_1 p} = -\epsilon'' \left(\frac{\partial F}{\partial x_1} \right) \quad (-18)$$

Now the overall effect becomes

$$\overline{u_1 \left(p + \frac{u_1 u_1}{2} \right)} = -(\epsilon'' + \epsilon') \left(\frac{\partial F}{\partial x_1} \right) \quad (-19)$$

Normally ϵ' is a positive quantity. However, because of the generally negative correlation between pressure and kinetic energy, we expect ϵ'' to be usually negative and smaller than ϵ' . Hence $(\epsilon'' + \epsilon')$ should be positive but smaller than ϵ' alone. Moreover, ϵ'' and ϵ' are both of the same dimension as the kinematic eddy viscosity ϵ . Also ϵ' can be expected to be roughly of the same order of magnitude as ϵ . For these reasons, it is advantageous to write.

$$(\epsilon'' + \epsilon') = \gamma \epsilon \quad (-20)$$

where γ is not a dimensionless coefficient roughly of order of magnitude unity.

Hence finally we write

$$\overline{u_1 \left(p + \frac{u_1 u_1}{2} \right)} = -\gamma \epsilon \left(\frac{\partial F}{\partial x_1} \right) \quad (-21)$$

Various numerical experiments suggest that the dimensionless empirical coefficient γ , like the coefficients α and f , depends only on the distance

y to the nearest fixed boundary, if any. The following expression has been found to give reasonably good results.

$$\gamma = 1.4 - 0.4e^{-\left(\frac{y}{\lambda} - 1\right)^2} \quad (-22)$$

Turning next to term V of the basic energy equation, the molecular diffusion effect, it may be said that this will normally be very much smaller than the other terms in the equation. It may safely be neglected everywhere except in the very thin laminar sublayer region along any solid boundary. We shall ignore it in our model.

Finally we combine the foregoing steps to obtain an overall heuristic energy equation as follows. Let

$$q^2 = 2E \quad (-23)$$

Then the energy equation becomes

$$\left(\frac{\partial E}{\partial t}\right) = \epsilon \Omega^{2-\beta} (2E)^{7/6} J^{1/3} + \frac{\partial}{\partial x_1} [U_1 E] + \frac{\partial}{\partial x_1} \left[\gamma \epsilon \left(\frac{\partial E}{\partial x_1}\right) \right] \quad (-24)$$

where ϵ , β , and α are fixed by the following empirical expressions, namely,

$$\frac{\epsilon}{\lambda J^{2E}} = \alpha = 0.065 \left\{ 1 - e^{-\left(\frac{y}{\lambda} - 1\right)^2} \right\} \quad (-25)$$

$$\beta = \frac{1}{3.7} \left\{ 1 + e^{-\left(\frac{y}{\lambda} - 1\right)^2} \right\}^{-1} \quad (-26)$$

$$\gamma = 1.4 - 0.4e^{-\left(\frac{y}{\lambda} - 1\right)^2} \quad (-27)$$

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11. Gawain, T. H. and Pritchett, J. W., "A Unified Heuristic Model of Fluid Turbulence," Journal of Computational Physics, Vol. 5, No. 3, June 1970.

The scientific literature on turbulence is very extensive, and no attempt will be made to list a comprehensive bibliography beyond the specific references cited above. For the reader interested in an overall orientation to the subject, the text books listed below might prove useful.

12. Batchelor, G. K., "Theory of Homogeneous Turbulence", Cambridge University Press, 1953.

13. Pai, S. I., "Viscous Flow Theory:", Van Nostrand, Frinceton, 1956.
14. Hinze, J. O., "Turbulence - An Introduction to Its Mechanism and Theory", McGraw Hill, 1959.

For specific accounts of more recent research, perhaps the two most useful journals are the "Journal of Fluid Mechanics" and the "Physics of Fluids."

APPENDICES

APPENDIX A

Derivation of Exact Energy Equation for Turbulence

Equation of Motion

Let prime marks on a symbol be used to denote the sum of the mean value of that quantity plus the turbulent perturbation. That is, let

$$U'_j = U_j + u_j \quad (A-1)$$

$$P' = P + p$$

Also let

$$T'_{ij} = T_{ij} + t_{ij} \quad (-2)$$

denote viscous shear stresses.

The equations of continuity and motion now become

$$\left(\frac{\partial U'_1}{\partial x_1} \right) = 0 \quad (-3)$$

$$\left(\frac{\partial U'_1}{\partial t} \right) = \left(\frac{\partial T'_{11}}{\partial x_1} \right) - \frac{\partial}{\partial x_1} (U'_1 U'_1) \quad (-4)$$

The corresponding energy equation is found by multiplying the equation of motion by U'_j . Thus

$$\frac{\partial}{\partial t} \left(\frac{U'_1 U'_1}{2} \right) = U'_j \left(\frac{\partial T'_{11}}{\partial x_1} \right) - U'_j \frac{\partial}{\partial x_1} (U'_1 U'_1) \quad (-5)$$

Now we substitute Eqs. (-1) and (-2) into (-3), (-4) and (-5), then expand, average and simplify. All terms linear in perturbation quantities vanish in the averaging. Therefore, the averaged equations of continuity, motion and energy become, respectively,

$$\left(\frac{\partial u_1}{\partial x_1}\right) = 0 \quad (-6)$$

$$\left(\frac{\partial u_1}{\partial t}\right) = \left(\frac{\partial T_{11}}{\partial x_1}\right) - \frac{\partial}{\partial x_1}(u_1 u_1 + \overline{u_1 u_1}) \quad (-7)$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{u_1 u_1}{2} + \frac{\overline{u_1 u_1}}{2}\right) &= u_1 \left(\frac{\partial T_{11}}{\partial x_1}\right) + \overline{u_1 \left(\frac{\partial T_{11}}{\partial x_1}\right)} + u_1 \frac{\partial}{\partial x_1}(u_1 u_1 + \overline{u_1 u_1}) \\ &\quad - u_1 \frac{\partial}{\partial x_1}\left(\frac{\overline{u_1 u_1}}{2}\right) - \overline{u_1 u_1} \left(\frac{\partial u_1}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(\frac{\overline{u_1 u_1 u_1}}{2}\right) \end{aligned} \quad (-8)$$

Also, multiplying the averaged equation of motion (-7) through by u_1 gives the energy equation for the mean flow, namely,

$$\frac{\partial}{\partial t}\left(\frac{u_1 u_1}{2}\right) = u_1 \left(\frac{\partial T_{11}}{\partial x_1}\right) - u_1 \frac{\partial}{\partial x_1}(u_1 u_1 + \overline{u_1 u_1}) \quad (-9)$$

Now subtracting (-10) from (-9) gives the energy equation for the turbulence itself, that is

$$\frac{\partial}{\partial t}\left(\frac{\overline{u_1 u_1}}{2}\right) = \overline{u_1 \left(\frac{\partial T_{11}}{\partial x_1}\right)} - \frac{\partial}{\partial x_1}\left[u_1 \frac{\overline{u_1 u_1}}{2}\right] - \overline{u_1 u_1} \left(\frac{\partial u_1}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(\frac{\overline{u_1 u_1 u_1}}{2}\right) \quad (-10)$$

It is convenient to symbolize the mean turbulent kinetic energy, the instantaneous kinetic energy, and the Reynolds stresses, respectively, by

$$E = \frac{\overline{u_1 u_1}}{2} \quad (-11)$$

$$E' = \frac{u_1 u_1}{2} \quad (-12)$$

$$\tau_{1j} = -\overline{u_1 u_j} \quad (-13)$$

Therefore the energy equation becomes

$$\left(\frac{\partial E}{\partial t}\right) = u_1 \left(\frac{\partial T_{11}}{\partial x_1}\right) - \frac{\partial}{\partial x_1}(u_1 E) + \tau_{1j} \left(\frac{\partial u_1}{\partial x_j}\right) - \frac{\partial}{\partial x_1}(\overline{u_1 E'}) \quad (-14)$$

The viscous term can now be developed further as follows. Consider the identity

$$\overline{u_j \left(\frac{\partial \tau_{1j}}{\partial x_1} \right)} = \frac{\partial}{\partial x_1} (\overline{u_1 \tau_{1j}}) - \overline{\tau_{1j} \left(\frac{\partial u_1}{\partial x_1} \right)} \quad (-15)$$

Now the viscous stress perturbation may be written

$$\tau_{1j} = -p \delta_{1j} + \nu \gamma_{1j} \quad (-16)$$

where

$$\gamma_{1j} = \left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1} \right) \quad (-17)$$

and

$$\left(\frac{\partial u_1}{\partial x_1} \right) = 0 \quad (-18)$$

Therefore

$$\overline{u_1 \tau_{1j}} = \overline{-u_1 p} + \nu \left(\frac{\partial E}{\partial x_1} - \frac{\partial \tau_{1j}}{\partial x_j} \right) \quad (-19)$$

$$\overline{\tau_{1j} \left(\frac{\partial u_1}{\partial x_1} \right)} = 0 + \nu \overline{\gamma_{1j} \left(\frac{\partial u_1}{\partial x_1} \right)} \quad (-20)$$

Substituting (-19) and (-20) into (-15) gives

$$\overline{u_j \left(\frac{\partial \tau_{1j}}{\partial x_1} \right)} = \frac{\partial}{\partial x_1} \left[\overline{-u_1 p} + \nu \left(\frac{\partial E}{\partial x_1} - \frac{\partial \tau_{1j}}{\partial x_j} \right) \right] - \nu \overline{\gamma_{1j} \left(\frac{\partial u_1}{\partial x_1} \right)} \quad (-21)$$

Substituting this back into (-14) and rearranging, we find

$$\begin{aligned} \left(\frac{\partial E}{\partial t} \right) = & \tau_{1j} \left(\frac{\partial u_1}{\partial x_1} \right) - \nu \overline{\gamma_{1j} \left(\frac{\partial u_1}{\partial x_1} \right)} - \frac{\partial}{\partial x_1} [u_1 E] \\ & - \frac{\partial}{\partial x_1} \left[u_1 \left(p + \frac{u_1 u_1}{2} \right) \right] + \frac{\partial}{\partial x_1} \left[\nu \left(\frac{\partial E}{\partial x_1} - \frac{\partial \tau_{1j}}{\partial x_j} \right) \right] \end{aligned} \quad (-22)$$

Two further simplifications are advantageous. Recall that

$$\Gamma_{ij} = \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \quad (-23)$$

Noting further that τ_{ij} and Γ_{ij} are both symmetrical tensors, we write

$$\tau_{ij} \Gamma_{ij} = \tau_{ij} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = 2\tau_{ij} \left(\frac{\partial u_i}{\partial x_j} \right) \quad (-24)$$

or

$$\tau_{ij} \left(\frac{\partial u_i}{\partial x_j} \right) = \frac{1}{2} \tau_{ij} \Gamma_{ij} \quad (-25)$$

By an exactly analogous argument we find that

$$\overline{\tau_{ij} \left(\frac{\partial u_i}{\partial x_j} \right)} = \frac{1}{2} \overline{\tau_{ij} \Gamma_{ij}} \quad (-26)$$

Consequently, the turbulence energy equation becomes finally,

$$\begin{aligned} \left(\frac{\partial E}{\partial t} \right) = & \frac{1}{2} \tau_{ij} \Gamma_{ij} - \frac{1}{2} \overline{v_{ij} \Gamma_{ij}} - \frac{\partial}{\partial x_i} (u_i E) - \frac{\partial}{\partial x_i} \left[u_i \left(p + \frac{u_i u_i}{2} \right) \right] \\ & + \frac{\partial}{\partial x_i} \left[v \left(\frac{\partial E}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} \right) \right] \end{aligned} \quad (-27)$$

It is important to notice that this equation is complete and exact. Each term has a definite physical significance as discussed in the main body of the report. The method of evaluating certain of the terms heuristically is also discussed in the main report.

APPENDIX B

Local Length Scale for Axi-Symmetric Mean Flows

Consider the following sequence of relations pertaining to the heuristic model, where the various integrals are summed over the entire flow field.

$$W = e^{-\frac{\Delta \bar{x} \cdot \Delta \bar{x}}{\lambda_1^2}} = \text{Weighting function} \quad (\text{B-1})$$

$$H = \int W dv' = \text{Normalizing integral} \quad (-2)$$

$$w = \frac{W}{H} = \text{Normalized weighting function} \quad (-3)$$

$$I^2 = \int w \Omega^4 dv' = \text{First characteristic integral} \quad (-4)$$

$$J^2 = \int w (\Omega \Omega')^2 dv' = \text{Second characteristic integral} \quad (-5)$$

$$\lambda^2 = \frac{I^2}{J^2} = \text{Local scale parameter} \quad (-6)$$

For axi-symmetric flows, the volume integrals (-2), (-4), and (-5) reduce to surface integrals over the meridional plane.

In this case the volume element may be written in the form

$$dv' = r' d\theta' dr' dz'$$

Also in cartesian coordinates

$$\Delta \bar{x} \cdot \Delta \bar{x} = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \quad (-7)$$

We may express the coordinates of the fixed point \bar{x} and of the variable point \bar{x}' in terms of cylindrical coordinates as follows.

$$\begin{aligned}
x &= 0 & x' &= r' \sin \theta' \\
y &= r & y' &= r' \cos \theta' \\
z &= z & z' &= z'
\end{aligned} \tag{-8}$$

Upon substituting (-8) into (-7) and rearranging, we can write

$$\Delta R \cdot \Delta R = \xi^2 + 2rr'(1 - \cos \theta') \tag{-9}$$

where

$$\xi^2 = (r' - r)^2 + (z' - z)^2 \tag{-10}$$

Now let us consider the typical integral I^2 as defined by Eq. (-4).

By making use of the foregoing relations, this integral can be reduced to the required form. In addition, it is convenient to utilize the following auxiliary variable, namely,

$$\zeta^2 = \frac{rr'}{\lambda_1^2} \tag{-11}$$

Eq. (-4) is now expressible in the form

$$I^2 = \frac{1}{r} \iint \left\{ e^{-\left(\frac{\xi}{\lambda_1}\right)^2} \left[\frac{\xi}{\sqrt{\pi}} \int_0^{2\pi} e^{2\zeta^2(1 - \cos \theta')} d\theta' \right] \sqrt{r'} \Omega \right\} dr' dz' \tag{-12}$$

Now since Ω is a function of z' and r' only, but independent of θ' , the above integration with respect to θ' can be made once and for all. This leads naturally to the definition of the auxiliary function

$$\psi(\zeta) = \frac{\xi}{\sqrt{\pi}} \int_0^{2\pi} e^{-2\zeta^2(1 - \cos \theta')} d\theta' \tag{-13}$$

The function ψ defined above must be computed numerically. The factor $\frac{\xi}{\sqrt{\pi}}$ was inserted on the right so as to achieve the simplification that at large values of ζ the function ψ approaches the limit unity, that is, that

$$\lim_{\zeta \rightarrow \infty} \psi = 1 \quad (-14)$$

It then happens that for small values of ζ

$$\lim_{\zeta \rightarrow 0} \psi = 2\sqrt{\pi} \zeta \quad (-15)$$

The function ψ is shown in Fig. B-1. Computed values are listed in Table B-1.

Once the function ψ is known, the integral I^2 may be rewritten as follows.

$$I^2 = \frac{1}{H} \sqrt{\frac{\pi}{r}} \iint e^{-\left(\frac{\xi}{\lambda_1}\right)^2} \psi\left(\frac{\sqrt{rr'}}{\lambda_1}\right) \sqrt{r'} \Omega^4 dr' dz' \quad (-16)$$

Similarly

$$J^2 = \frac{1}{H} \sqrt{\frac{\pi}{r}} \iint e^{-\left(\frac{\xi}{\lambda_1}\right)^2} \psi\left(\frac{\sqrt{rr'}}{\lambda_1}\right) \sqrt{r'} (\Omega^2 \Omega'^2) dr' dz' \quad (-17)$$

The scale parameter λ^2 is now found from (-6). Notice that in computing λ^2 from (-6) the factors $\frac{1}{H} \sqrt{\frac{\pi}{r}}$ in (-16) and (-17) cancel from the result.

Eqs. (-16) and (-17) illustrate how the characteristic integrals reduce from volume to surface integrals for the axi-symmetric case. Eqs. (-11) and (-13) define ψ which is a function characteristic of axi-symmetric mean flows.

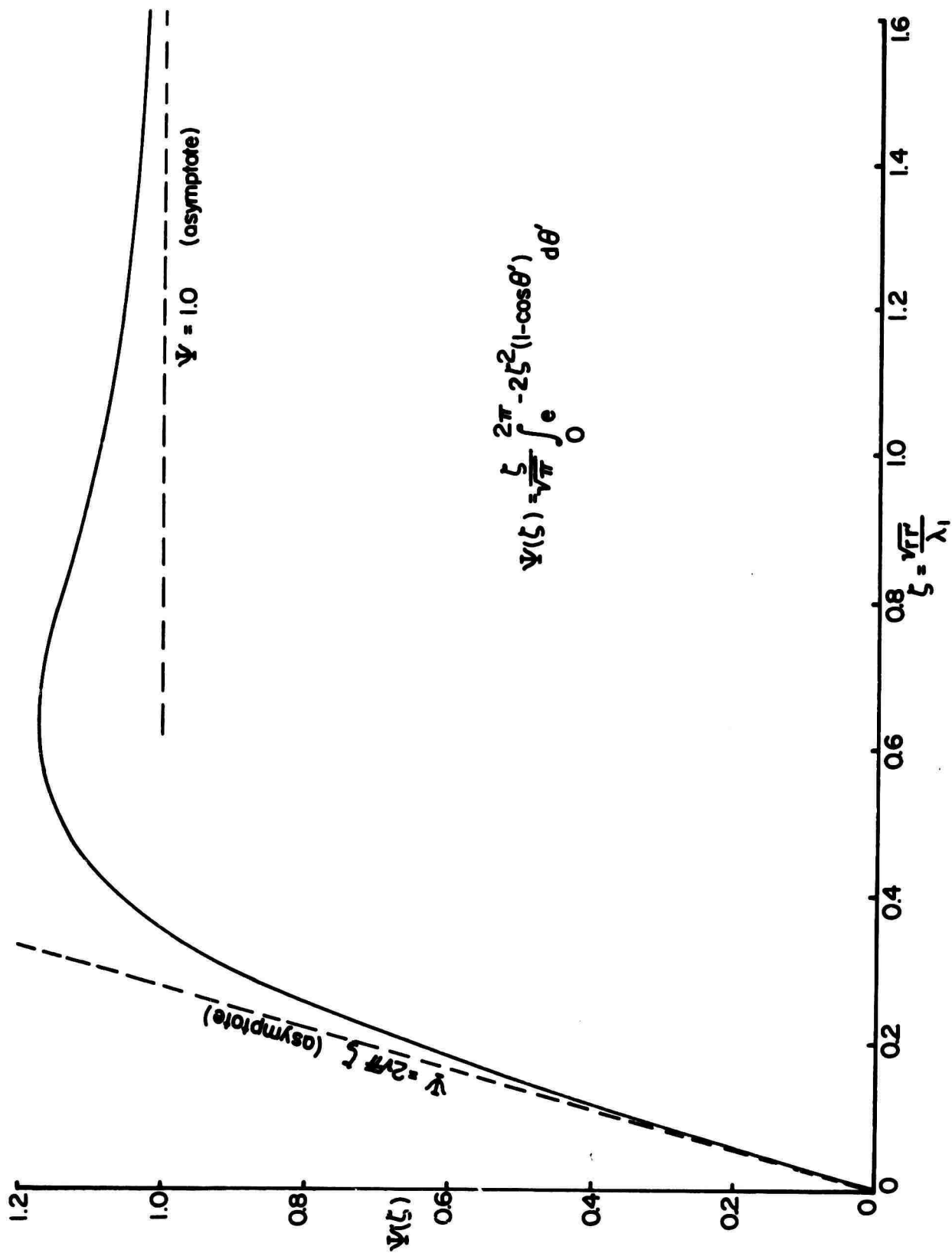


FIGURE B-1 . GRAPH OF THE FUNCTION $\Psi(\zeta)$

TABLE B-1

COMPUTED VALUES OF THE ψ FUNCTION

(Refer to Eqs. B-11, B-13)

0.0	0.0000
0.1	0.3475
0.2	0.6555
0.3	0.8955
0.4	1.0562
0.5	1.1433
0.6	1.1739
0.7	1.1687
0.8	1.1461
0.9	1.1187
1.0	1.0936
1.2	1.0581
1.4	1.0387
1.6	1.0279
1.8	1.0213
2.0	1.0169
2.5	1.0105
3.0	1.0072
4.0	1.0040
5.0	1.0025
∞	1.0000

APPENDIX C

Experimental Data

Various useful reference items of experimental data are summarized in Tables C-1 through C-5 and in Figs. C-1 and C-2. These tabulations and diagrams are largely self-explanatory.

Table C-2 is of special interest in that it shows a comparison between theoretical values of dissipation, as computed from the heuristic model, and the experimental measurements of Laufer, Ref. (10). The agreement is satisfactory.

Laufer's measured values are known to be too low and he attempted to estimate better values as shown.

Shearing rates were estimated on the basis of a theoretical mean velocity profile using Prandtl's mixing length theory and Nikuradse's empirical curve for the mixing length l , namely,

$$l = 0.14 - 0.08r^2 - 0.06r^4 \quad (C-1)$$

The information in Table C-2 is also displayed graphically in Figs. C-1 and C-2. There is no particular significance to the fact that the ordinate in Fig C-2 is $\frac{1}{\sqrt{\beta}}$ rather than β itself; this merely reflects the circumstance that a slightly different notation was being employed at the time these calculations were actually made than that which is used in this report.

In view of the uncertainties in the experimental data itself, the agreement between theory and experiment as revealed in Figs. C-1 and C-2 is regarded as satisfactory.

The remaining tables require no particular comment except, perhaps, to note that the available data is meagre and not very satisfactory. Note that the data given is mostly one dimensional, with slight information in the second dimension and none at all in the third. There is clearly a need for more experimental work.

TABLE C-1

TURBULENT ENERGY DATA FOR A PIPE

Measurements by Laufer, Ref. (10)

All quantities non-dimensionalized using radius $a = 1$ and friction velocity $v^* = 1$.

r	<u>$R_e = 500,000$</u>		<u>$R_e = 50,000$</u>		
	2E	\dot{E}_H	2E	\dot{E}_H	$\frac{1}{2} \tau_{1j} \Gamma_{1j}$
0	1.65	2.1	1.69	2.15	--
0.1	1.80	2.2	1.80	2.26	--
0.2	2.05	2.4	2.06	2.72	--
0.3	2.45	3.1	2.41	3.28	1.5
0.4	2.90	3.8	2.85	3.96	2.5
0.5	3.50	4.8	3.48	4.76	3.7
0.6	4.20	6.2	4.01	6.23	5.3
0.7	5.00	8.7	4.49	8.10	7.6
0.8	6.00	13.0	5.13	11.3	11.7
0.9	7.40	22.8	5.66	20.9	22.0
0.98	--	--	7.85(peak)	--	--

TABLE C-2

MEASURED AND COMPUTED ENERGY DISSIPATIONRATES IN A PIPE AT $R_e = 500,000$

Measurements by Laufer, Ref. (10)

All quantities non-dimensionalized using radius $a = 1$ and friction velocity $v^* = 1$.

r	$(2E)^{7/6}$	$(\Omega\Omega')^{1/3}$	β^{-1}	\dot{E}_H		
	Table C-1	Calc. * Eq. (1.3-18)	Eq. (2.8-14)	Meas.	Est.	
0	1.78	2.94	3.70	1.43	1.0	2.0
0.1	2.00	2.98	3.70	1.61		
0.2	2.45	3.09	4.58	1.65	1.2	2.5
0.3	2.95	3.29	6.71	1.45		
0.4	3.45	3.62	7.35	1.71	2.0	4.0
0.5	4.30	4.16	7.40	2.42		
0.6	5.30	5.03	7.40	3.61	3.0	6.0
0.7	6.70	6.52	7.40	5.91		
0.8	8.20	9.58	7.40	10.60	7.0	13.5
0.9	10.00	18.80	7.40	25.40	15.0	25.0

Notes: 1. Laufer regards his measured dissipation data as too low and attempts to estimate better values as indicated.

2. The quantity $(\Omega\Omega')^{1/3}$ was obtained from the theoretical velocity profile based on the Prandtl-Nikuradse mixing length theory using Eq. (C-1) Refer also to Eq. (1.3-7).

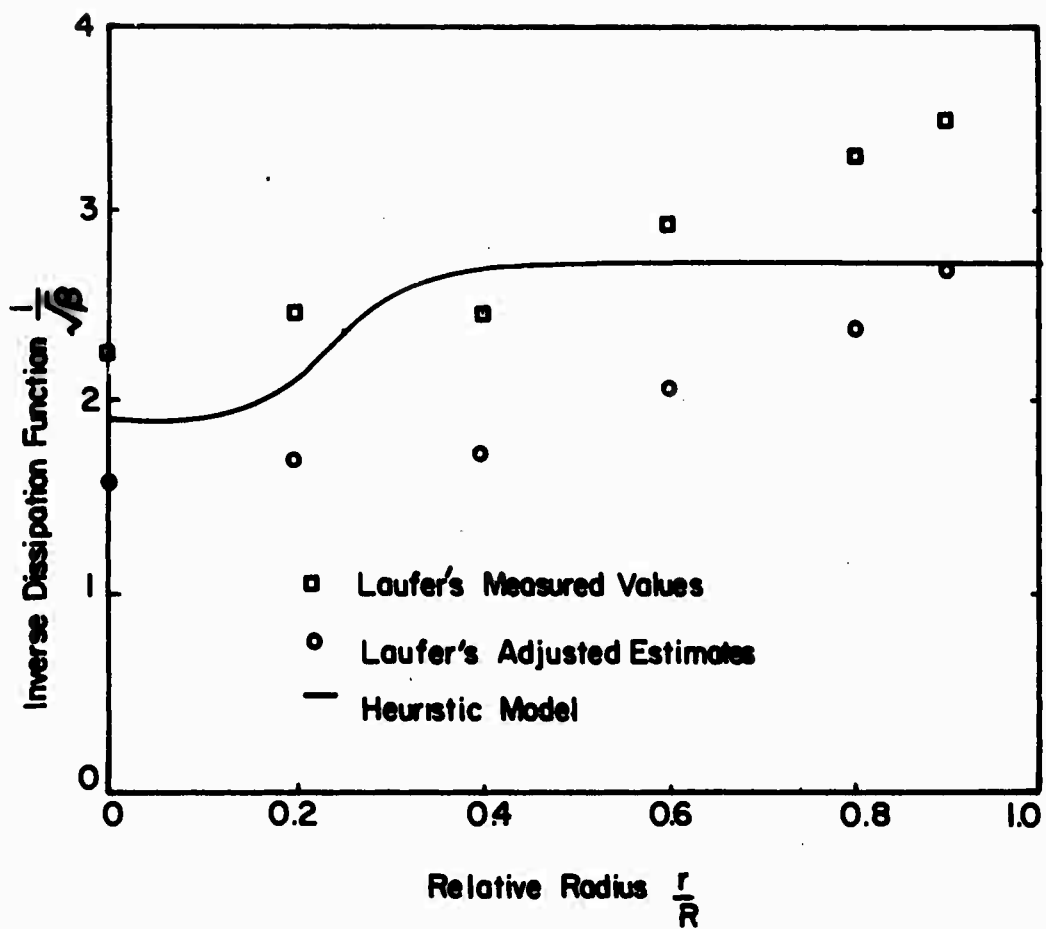


FIGURE C-1 . INVERSE DISSIPATION FUNCTION $\frac{1}{\sqrt{\beta}}$
IN A PIPE

TABLE C-3

AXIAL VELOCITIES IN A COLD TURBULENT JET

AT AXIAL STATION $\frac{z}{D} = 20$

Measurement by Corrsin and Uberoi, Ref. (7)

- V = mean axial velocity
 V_o = axial velocity at axis
 v = root mean square axial perturbation
 $b_{1/2}$ = radius to point where $\frac{v}{V_o} = \frac{1}{2}$
 z = axial coordinate
 D = diameter of jet

$\left(\frac{r}{b_{1/2}}\right)$	$\left(\frac{v}{V_o}\right)$	$\left(\frac{v}{V_o}\right)$
0	1.0	0.205
0.2	0.95	0.21
0.4	0.85	0.215
0.6	0.74	0.215
0.8	0.625	0.20
1.0	0.500	0.18
1.2	0.40	0.165
1.4	0.32	0.145
1.6	0.25	0.12
1.8	0.18	0.09
2.0	0.135	0.07
2.2	0.08	0.045
2.4	0.055	0.03
2.6	0.04	0.02

TABLE C-4

AXIAL AND RADIAL VELOCITIES IN A HEATED TURBULENT JET

AT AXIAL STATION $\frac{z}{D} - 15$

Measurements by Corrsin and Uberoi, Ref. (7)

u = root mean square radial perturbation

Other notation as in Table C-3

$\frac{r}{b_{1/2}}$	$\left(\frac{v}{v_o}\right)$	$\left(\frac{v}{v_o}\right)$	$\left(\frac{u}{v_o}\right)$
0	1.0	0.17	0.175
0.2	0.97	0.17	0.17
0.4	0.88	0.17	0.16
0.6	0.76	0.17	0.14
0.8	0.62	0.17	0.12
1.0	0.50	0.15	0.09
1.2	0.38	0.13	0.075
1.4	0.28	0.10	0.05
1.6	0.205	0.075	0.035
1.8	0.145	0.05	0.022
2.0	0.10	0.03	0.013
2.2	0.055	0.02	0.005
2.4	0.025	0.01	0

TABLE C-5

AXIAL VELOCITIES IN A TURBULENT JET

AT TWO AXIAL STATIONS

Measurements by Laurence, Ref. (8)

Notation as in Tables C-3 and C-4

$\frac{y}{r} = \eta$	At $\frac{z}{D} = 16$		At $\frac{z}{D} = 20$	
	$\left(\frac{v}{v_o}\right)$	$\left(\frac{v}{v_o}\right)$	$\left(\frac{v}{v_o}\right)$	$\left(\frac{v}{v_o}\right)$
0	0.43	0.082	0.36	0.073
0.5	--	0.082	--	0.073
1.0	0.40	0.082	0.34	0.073
1.5	--	0.084	--	0.074
2.0	0.31	0.089	0.30	0.075
2.5	--	0.091	--	0.075
3.0	0.22	0.090	0.25	0.075
3.5	--	0.084	--	0.075
4.0	0.15	0.074	0.19	0.075
4.5	--	0.061	--	0.072
5.0	0.07	0.047	0.135	0.067
6.0			0.08	
7.0			0.025	
8.0			0	